On the Hurwitz Function for Rational Arguments

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Abstract

Using functional properties of the Hurwitz zeta function and symbolic derivatives of the trigonometric functions, the function $\zeta(2n+1, p/q)$ is expressed in several ways in terms of other mathematical functions and numbers, including in particular the Glaisher numbers.

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1 Introduction

The Hurwitz zeta function, one of the fundamental transcendental functions, is traditionally defined (see [3, 4, 15, 17]) by the series

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad \Re(s) > 1, \quad a \notin \mathbb{Z}_0^- \tag{1}$$

The series converges absolutely for $\Re(s) = \sigma > 1$ and the convergence is uniform in every half-plane $\sigma \geq 1 + \delta$ ($\delta > 0$). Therefore, $\zeta(s,a)$ is analytic function of s in the half-plane $\Re(s) = \sigma > 1$. It satisfies the following integral representation (see [3], theorem 12.2)

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}}\, dx, \quad \Re(s) > 1, \quad \Re(a) > 0 \tag{2}$$

Using the Taylor expansion $1/(1-e^x) = -\frac{1}{x} + \frac{1}{2} + O(x)$ and splitting (2) into two integrals

$$\begin{aligned}\zeta(s,a+1) &= \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\left(\frac{1}{1-e^x}-\frac{1}{x}+\frac{1}{2}\right)dx \\ &+\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\left(\frac{1}{x}-\frac{1}{2}\right)dx\end{aligned}$$

we can analytically continue $\zeta(s,a)$ into the strip $-1 < \Re(s) < 1$. Evaluating the second integral, replacing a by $a+1$ and making use of

$$\zeta(s,a+1) = \zeta(s,a) - \frac{1}{a^s}, \quad \Re(a) > 0 \tag{3}$$

we obtain

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} + \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\left(\frac{1}{1-e^x}-\frac{1}{x}+\frac{1}{2}\right)dx, \tag{4}$$
$$\Re(s) > -1, \Re(a) > 0$$

This could be further generalized to (see [17, p.93])

$$\zeta(s,a) = a^{-s} + \sum_{k=0}^{n}\frac{\Gamma(k+s-1)}{\Gamma(s)}\frac{B_k}{k}\,a^{1-s-k} +$$
$$\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}\left(\frac{1}{1-e^x}-\sum_{k=0}^{n}\frac{B_k}{k}\,a^{1-k}\right)dx, \tag{5}$$
$$\Re(s) > -2\lfloor\frac{n}{2}\rfloor - 1, \Re(a) > 0$$

where B_k are the Bernoulli numbers.

We can also analytically continue the Hurwitz zeta function to the whole complex s-plane (except for a simple pole at $s=1$) by means of the contour integral

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{x^{s-1}e^{ax}}{1-e^x}\,dx \tag{6}$$

where the contour C is a loop that starts from $-\infty$ along the lower side of the real axis, encircle the origin and then returns to $-\infty$ along the upper side of the real axis.

Among all integral representations for $\zeta(s,a)$, the Hermite integral has a definite place. Its derivation is based on the summation formula of Plana (see [4, p.22] or [17, p.90])

$$\sum_{k=0}^{\infty} f(k) = \frac{1}{2}f(0) + \int_0^\infty f(x)\,dx + i\int_0^\infty \frac{f(ix)-f(-ix)}{e^{2\pi x}-1}\,dx \tag{7}$$

Choosing $f(k)$ in (7) as

$$f(k) = \frac{1}{(k+a)^s}$$

after some obvious evaluations, we readily find

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + i\int_0^\infty \frac{(a+ix)^{-s} - (a-ix)^{-s}}{e^{2\pi x}-1}\,dx, \qquad \Re(a) > 0 \tag{8}$$

Noting that

$$(a+ix)^{-s} - (a-ix)^{-s} = \frac{2}{i\,(a^2+x^2)^{s/2}} \sin(s \arctan(\frac{x}{a})) \tag{9}$$

we rewrite (8) in an equivalent form

$$\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin\left(s \arctan\left(\frac{x}{a}\right)\right)}{(x^2+a^2)^{s/2}(e^{2\pi x}-1)}\,dx, \qquad s \neq 1, \Re(a) > 0 \tag{10}$$

This integral is known as Hermite's integral of $\zeta(s,z)$.

The Riemann zeta function $\zeta(s)$ is a special case of the Hurwitz zeta function

$$\zeta(s,1) = \zeta(s)$$

Many series and integral representations for $\zeta(s)$ follows straightforwardly from correspondent representations for the Hurwitz zeta function. The same as the latter, the Riemann zeta function is an analytic function in the whole complex s-plane, except the pole at $s = 1$.

Another special case of the Hurwitz function is the Bernoulli polynomials

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}, \quad n \in \mathrm{N}_0 \tag{11}$$

This formula follows directly from integral (6) by means of the residue theorem

$$\zeta(-n,a) = \frac{\Gamma(n+1)}{2\pi i}\int_C \frac{x^{-n-1}e^{ax}}{1-e^x}\,dx = -n!\,\operatorname*{Res}_{x=0}\left(\frac{1}{x^{n+2}}\frac{xe^{ax}}{e^x-1}\right) \tag{12}$$

and the generating function for the Bernoulli numbers

$$\frac{xe^{ax}}{e^x-1} = \sum_{k=0}^\infty B_n(a)\frac{x^n}{n!} \tag{13}$$

The Hurwitz function is closely related to the multiple gamma function $\Gamma_n(z)$ defined as a generalization of the classical Euler gamma function $\Gamma(z)$, by the following recurrence-functional equation (for references and further reading see [1, 2, 5, 17]):

$$\begin{aligned} \Gamma_{n+1}(z+1) &= \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad z \in \mathrm{C}, \quad n \in \mathrm{N}, \\ \Gamma_1(z) &= \Gamma(z), \\ \Gamma_n(1) &= 1. \end{aligned} \tag{14}$$

The simplest functional relationship between $\zeta(t,z)$ and $\Gamma_n(z)$ is given by (we refer the reader to [1] for other relations of this type)

$$\log\Gamma_2(z+1) + z\,\log\Gamma(z) = \zeta'(-1,z) - \zeta'(-1), \quad \Re(z) > 0, \tag{15}$$

where $\zeta'(t,z) = \frac{d}{dt}\zeta(t,z)$ is defined as an analytic continuation provided by the Hermite integral (10):

$$\begin{gathered} \zeta'(-1,z) = \frac{z^2}{2}\log z - \frac{z^2}{4} - \frac{z}{2}\log z + \int_0^\infty \frac{2\,z\arctan(\frac{x}{z}) + x\,\log(x^2+z^2)}{e^{2\pi x}-1}\,dx, \\ \Re(z) > 0. \end{gathered} \tag{16}$$

The constant $\zeta'(-1) = \zeta'(-1,0)$ is known as the Glaisher-Kinkelin constant A, defined by

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{17}$$

It satisfies the following integral representation

$$\log A = \frac{1+\log(2\pi)}{12} - \frac{1}{2\pi^2}\int_0^\infty \frac{x\log x}{e^x-1}\,dx. \tag{18}$$

The constant A originally appeared in papers by Kinkelin [11] and Glaisher [7, 8] on the asymptotic expansion when $n \to \infty$ of the following product

$$1^{1^p}\,2^{2^p}\,\ldots n^{n^p}, \quad p \in \mathrm{N}.$$

For $p = 1$, we have

$$1^1\,2^2\,\ldots n^n = n!^n\,\Gamma_2(n+1) = A\,\exp\left(\zeta'(-1,n+1) - \frac{1}{12}\right) \tag{19}$$

Generally,

$$1^{1^p}\,2^{2^p}\,\ldots n^{n^p} = \exp\Big(\zeta'(-p,n+1) - \zeta'(-p)\Big) \tag{20}$$

From the integral (6) one can derive the Hurwitz series representation

$$\zeta(s,a) = \frac{\Gamma(1-s)\, i}{(2\pi)^{1-s}} \left(e^{-\pi i s/2} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{1-s}} - e^{\pi i s/2} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{1-s}} \right) \tag{21}$$

valid in the half-plane $\Re(s) < 0$ and $0 < a < 1$. If $a \neq 1$ this representation is also valid for $\Re(s) < 1$. The Dirichlet series in (21) can be rewritten in terms of the polylogarithm $\mathrm{Li}_m(z)$:

$$\zeta(s,a) = \frac{\Gamma(1-s)\, i}{(2\pi)^{1-s}} \left(e^{-\pi i s/2} \mathrm{Li}_{1-s}(e^{2\pi i a}) - e^{\pi i s/2} \mathrm{Li}_{1-s}(e^{-2\pi i a}) \right) \tag{22}$$

or, equivalently, (for $\Re(s) > 1$)

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^{s}} \left(e^{-\pi i s/2} \mathrm{Li}_{s}(e^{-2\pi i a}) + e^{\pi i s/2} \mathrm{Li}_{s}(e^{2\pi i a}) \right) \tag{23}$$

Setting $a = 1$, yields the functional equation for the Riemann zeta function

$$\zeta(1-s) = 2\frac{\Gamma(s)}{(2\pi)^{s}} \cos(\frac{\pi s}{2})\zeta(s) \tag{24}$$

The result holds for all admissible s by analytic continuation.

It is easy to see that the polylogarithm $\mathrm{Li}_s(e^{2\pi i a})$ is a linear combination of the Hurwitz zeta functions when parameter a is rational. Indeed, rearranging terms according to the residue classes mod q, we obtain

$$\begin{aligned} \mathrm{Li}_s(e^{2\pi i p/q}) &= \sum_{n=1}^{q} \sum_{k=0}^{\infty} \frac{e^{2\pi i n p/q}}{(qk+n)^s} = \sum_{n=1}^{q} e^{2\pi i n p/q} \sum_{k=0}^{\infty} \frac{1}{(qk+n)^s} \\ &= \frac{1}{q^s} \sum_{n=1}^{q} e^{2\pi i n p/q} \zeta(s, \frac{n}{q}) \end{aligned}$$

Now replacing the polylogarithms in (23) by the above finite sum, we obtain

$$\zeta\big(1-s, \frac{p}{q}\big) = \frac{2\Gamma(s)}{(2\pi q)^s} \sum_{k=1}^{q} \cos\left(\frac{\pi s}{2} - \frac{2\pi k p}{q}\right) \zeta\big(s, \frac{k}{q}\big), \quad 1 \leq p \leq q \tag{25}$$

which holds true for all $s \neq 0$.

The main object of this paper is to derive a family of closed form representations for $\zeta(2n+1, \frac{p}{q})$ by evaluating symbolic derivatives of certain trigonometric functions. We will also show how the results presented here relate to those that were obtained in other works.

2 Reflection formula

In this section we express symbolic derivatives of the cotangent function in a closed form. Simple observation

$$\frac{d}{dz}\cot(z) = -(1+\cot^2(z)$$

$$\frac{d^2}{dz^2}\cot(z) = 2\cot(z)(1+\cot^2(z)$$

suggests that $\frac{d^n}{dz^n}\cot(z)$, $n \in \mathrm{N}$ is a polynomial in $\cot(z)$. Generally, some trigonometric and hyperbolic functions fall to the class of functions whose derivatives are polynomials in terms of the same function (see [10, 12, 13]).

Lemma 2.1 *For any integer* $n > 1$,

$$\left(\frac{d}{dz}\right)^n \cot(z) = (2i)^n(\cot(z)-i)\sum_{j=1}^{n}\frac{k!}{2^k}\left\{ n \atop k \right\}(i\cot(z)-1)^k \tag{26}$$

where $\left\{ j \atop k \right\}$ *are the Stirling subset numbers, defined by [16]*

$$\left\{ n \atop k \right\} = k\left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}, \quad \left\{ n \atop 0 \right\} = \delta_n \tag{27}$$

Proof. We start with the identity

$$\cot(z) = i\left(1+\frac{2}{e^{2iz}-1}\right)$$

Setting $e^{iz} \to y$, we have

$$\left(\frac{d}{dz}\right)^n \cot(z) \overset{e^{iz}\to y}{=} i^{n+1}\left(y\frac{d}{dy}\right)^n\left(1+\frac{2}{y^2-1}\right) = i^{n+1}\left(y\frac{d}{dy}\right)^n\frac{2}{y^2-1}$$

Expanding $\frac{1}{y^2-1}$ into a geometric series and then differentiating, we obtain

$$\left(\frac{d}{dz}\right)^n \cot(z) \overset{e^{iz}\to y}{=} -(2i)^{n+1}\sum_{k=1}^{\infty}k^n y^{2k}$$

Next, we make a use of (see [12])

$$\sum_{k=1}^{n}k!\left\{ n \atop k \right\}z^k = \frac{1}{1+z}\sum_{k=1}^{\infty}k^n\left(\frac{z}{1+z}\right)^k, \quad n\in\mathrm{N} \quad |z|<1 \tag{28}$$

which is proved by means of the generating function

$$k^n = \sum_{j=1}^{n} (-1)^j (-k)_j \left\{ {n \atop j} \right\} \tag{29}$$

where $(k)_j$ is the Pochhammer symbol. Substituting (29) into the right hand side of (28), yields

$$\frac{1}{z+1} \sum_{k=1}^{\infty} \sum_{j=1}^{n} (-1)^j (-k)_j \left\{ {n \atop j} \right\} \left(\frac{z}{z+1} \right)^k$$

Changing the order of summation and evaluating the inner sum

$$\sum_{k=1}^{\infty} (-k)_m y^k = \frac{m!}{1-y} \left(\frac{y}{y-1} \right)^m$$

proves (28).

Therefore, in a view of (28), the derivative of the cotangent immediately reduces to

$$\left(\frac{d}{dz} \right)^n \cot(z) \overset{e^{iz} \to y}{=} \frac{(2i)^{n+1}}{y^2-1} \sum_{j=1}^{n} k! \left\{ {n \atop k} \right\} \left(\frac{y^2}{1-y^2} \right)^k$$

Finally, we set $y = e^{iz}$ into the right hand side to obtain formula (26). □

In the similar way we can derive a representation for the derivatives of the tangent

$$\left(\frac{d}{dz} \right)^n \tan(z) = (-2i)^n (\tan(z) - i) \sum_{j=1}^{n} \frac{k!}{2^k} \left\{ {n \atop k} \right\} (i \tan(z) - 1)^k \tag{30}$$

To complete the picture, by changing $z \to iz$, we obtain representations for hyperbolic functions

$$\left(\frac{d}{dz} \right)^n \coth(z) = 2^n (\coth(z) - 1) \sum_{j=1}^{n} \frac{(-1)^k k!}{2^k} \left\{ {n \atop k} \right\} (\coth(z) + 1)^k \tag{31}$$

$$\left(\frac{d}{dz} \right)^n \tanh(z) = 2^n (\tanh(z) - 1) \sum_{j=1}^{n} \frac{(-1)^k k!}{2^k} \left\{ {n \atop k} \right\} (\tanh(z) + 1)^k \tag{32}$$

Theorem 2.2 *For any integer $n > 1$ and $0 < x < 1$,*

$$\zeta(n, 1-x) + (-1)^n \zeta(n, x) =$$

$$\frac{(2\pi i)^n}{2(n-1)!} \big(i \cot(\pi x) + 1 \big) \sum_{k=1}^{n-1} \frac{k!}{2^k} \left\{ {n-1 \atop k} \right\} \big(i \cot(\pi x) - 1 \big)^k \tag{33}$$

where $\left\{ {j \atop k} \right\}$ are the Stirling subset numbers.

Proof. The result follows from the reflection formula for the Hurwitz function

$$\zeta(n,1-x)+(-1)^n\zeta(n,x)=-\frac{\pi}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\cot(\pi x),\quad n>1,\quad 0<x<1 \tag{34}$$

and Lemma 2.1. ☐

Theorem 2.3 *For $n,p,q\in \mathrm{N}$ and $1\le p\le q$,*

$$\zeta(2n+1,1-\frac{p}{q})-\zeta(2n+1,\frac{p}{q})=-\frac{\pi^{2n+1}}{(2n)!}\lim_{y\to 0}\left(\frac{d}{dy}\right)^{2n}\frac{\sin(2\pi p/q)}{\cos(2y)-\cos(2\pi p/q)} \tag{35}$$

Proof. We replace n by $2n+1$ and x by p/q in formula (33) and then use identity (28) to obtain

$$\zeta(2n+1,1-\frac{p}{q})-\zeta(2n+1,\frac{p}{q})=\frac{i(-1)^n(2\pi)^{2n+1}}{(2n)!}\lim_{y\to\exp(\frac{\pi ip}{q})}\sum_{k=1}^{\infty}k^{2n}y^{2k}$$

where the series is understood as a meromorphic function in $y\in \mathrm{C}$

$$\sum_{k=1}^{\infty}k^{2n}y^{2k}=-\frac{1}{2^{2n}}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}$$

Hence,

$$\zeta(2n+1,1-\frac{p}{q})-\zeta(2n+1,\frac{p}{q})=-\frac{i(-1)^n(2\pi)^{2n+1}}{2^{2n}(2n)!}\lim_{y\to\exp(\frac{\pi ip}{q})}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1} \tag{36}$$

It is easy to verify by direct evaluation that

$$\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}=-\left(z\frac{d}{dz}\right)^{2n}\frac{1}{z^2-1}\bigg|_{z=\frac{1}{y}}$$

for $n=1,2,\ldots$ and, therefore,

$$\lim_{y\to\exp(\frac{\pi ip}{q})}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}=-\lim_{y\to\exp(-\frac{\pi ip}{q})}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}$$

Thus, formula (36) transforms to

$$\zeta(2n+1,1-\frac{p}{q})-\zeta(2n+1,\frac{p}{q})=-\frac{i(-1)^n\pi^{2n+1}}{(2n)!}\cdot$$

$$\left(\lim_{y\to\exp(\frac{\pi ip}{q})}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}-\lim_{y\to\exp(-\frac{\pi ip}{q})}\left(y\frac{d}{dy}\right)^{2n}\frac{1}{y^2-1}\right)$$

Replacing y by $y\,e^{\pi i p/q}$ and collecting terms, yields

$$\zeta(2n+1,1-\frac{p}{q})-\zeta(2n+1,\frac{p}{q})=-\frac{(-1)^n\pi^{2n+1}}{(2n)!}\lim_{y\to 1}\left(y\frac{d}{dy}\right)^{2n}\frac{2\sin(\frac{2\pi p}{q})y^2}{y^4-2\cos(\frac{2\pi p}{q})y^2+1}$$

which finally leads to (35) upon substitution $y\to\exp(iy)$. □

3 Special values of $\zeta(2n+1,p/q)$

In this section we consider special cases of $\zeta(2n+1,\frac{p}{q})$, $\frac{p}{q}=\frac{1}{6},\frac{5}{6},\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{2}{3},\frac{1}{2}$ when it is expressible in terms of other transcendental functions and constants. Originally this problem was solved by K.S. Kolbig [14], who expressed $\psi^{(2n)}(\frac{p}{q})$ in finite sums of the Bernoulli numbers, using functional properties of the polylogarithm. In our approach, $\zeta(2n+1,\frac{p}{q})$ are expressed in terms of generating functions of trigonometric functions.

The case $q=2$ is trivial and follows immediately from the multiplication formula (which, in turns, can be easily proved by rearranging terms in (1) according to the residue classes $\bmod n$)

$$\zeta(s,nz)=n^{-s}\sum_{k=0}^{n-1}\zeta(s,z+\frac{k}{n}),\quad n\in\mathrm{N} \tag{37}$$

Setting $z=1/2$ and $n=1$ in (37), we obtain

$$\zeta(s,\frac{1}{2})=(2^s-1)\,\zeta(s) \tag{38}$$

Next, we will derive other special cases of $\zeta(2n+1,p/q)$ based on the reflection formula (35).

Theorem 3.1 *For $n\in\mathrm{N}$,*

$$\left.\begin{matrix}\zeta(2n+1,\frac{1}{4})\\ \\ \zeta(2n+1,\frac{3}{4})\end{matrix}\right\}=2^{2n}\left(2^{2n+1}-1\right)\zeta(2n+1)\pm\frac{(-1)^n(2\pi)^{2n+1}}{4(2n)!}\,E_{2n} \tag{39}$$

where E_n are the Euler numbers.

Proof. From the multiplication formula (37) with $n=4, z=1/4$ and Theorem 2.3 with $p=1$, $q=4$ we find

$$\zeta(2n+1,\frac{1}{4})=2^{2n}\left(2^{2n+1}-1\right)\zeta(2n+1)+\frac{\pi^{2n+1}}{2(2n)!}\left(\frac{d}{dy}\right)^{2n}\sec(2y)\Big|_{y=0}$$

Recalling the generating function for the Euler numbers

$$\mathrm{sech}(z) = \sum_{k=0}^{\infty} E_n \frac{z^n}{n!}$$

we conclude the proof. □

Formula (39) along with (33) suggests a functional relation between the Stirling and Euler numbers.

Lemma 3.2 *For* $n \in \mathrm{N}$*,*

$$(1-i)\sum_{k=1}^{2n}\left(\frac{i-1}{2}\right)^k k!\left\{ {2n \atop k} \right\} = E_{2n} \tag{40}$$

$$(i+1)\sum_{k=1}^{2n}\left(-\frac{1+i}{2}\right)^k k!\left\{ {2n \atop k} \right\} = E_{2n} \tag{41}$$

where $i = \sqrt{-1}$*.*

Proof. We prove the first identity (40). By virtue of (28), we have

$$(1-i)\sum_{k=1}^{2n}\left(\frac{i-1}{2}\right)^k k!\left\{ {2n \atop k} \right\} = -2i \lim_{z\to i}\sum_{k=1}^{\infty} k^{2n} z^k$$

where the series is understood by the analytic continuation. Furthermore, since

$$\sum_{k=1}^{\infty} k^n z^k = \left(z\frac{d}{dz}\right)^n \frac{z}{1-z}$$

we have

$$(1-i)\sum_{k=1}^{2n}\left(\frac{i-1}{2}\right)^k k!\left\{ {2n \atop k} \right\} = -2i \lim_{z\to i}\left(z\frac{d}{dz}\right)^{2n} \frac{z}{1-z}$$

It is not difficult to verify the following chain of transformations

$$\begin{aligned}\lim_{z\to i}\left(z\frac{d}{dz}\right)^{2n}\frac{z}{1-z} &= \lim_{z\to i}\left(z\frac{d}{dz}\right)^{2n}\frac{1}{1-z} = \frac{1}{2}\lim_{z\to i}\left(z\frac{d}{dz}\right)^{2n}\left(\frac{1}{1-z}-\frac{1}{1+z}\right)\\ &= \lim_{z\to i}\left(z\frac{d}{dz}\right)^{2n}\frac{z}{1-z^2} = i\lim_{z\to 1}\left(z\frac{d}{dz}\right)^{2n}\frac{z}{1+z^2}\end{aligned}$$

Therefore,

$$(1-i)\sum_{k=1}^{2n}\left(\frac{i-1}{2}\right)^k k!\left\{\begin{matrix}2n\\k\end{matrix}\right\} = \lim_{z\to 1}\left(z\,\frac{d}{dz}\right)^{2n}\left(\frac{2z}{1+z^2}\right) \tag{42}$$

On the other hand,

$$E_{2n} = \lim_{z\to 0}\left(\frac{d}{dz}\right)^{2n}\operatorname{sech}(z) = \lim_{y\to 1}\left(y\,\frac{d}{dy}\right)^{2n}\frac{2y}{1+y^2} \tag{43}$$

where $y = e^z$. Comparing (42) with (43), concludes the proof. $\square$

Similarly to Theorem 3.1 we derive closed form representations for $q = 3$ and $q = 6$.

Theorem 3.3 *For $n \in \mathrm{N}$,*

$$\left.\begin{matrix}\zeta(2n+1,\frac{1}{6})\\ \\ \zeta(2n+1,\frac{5}{6})\end{matrix}\right\} = \frac{1}{2}(3^{2n+1}-1)(2^{2n+1}-1)\zeta(2n+1)$$
$$\pm\frac{\pi^{2n+1}\sqrt{3}}{2(2n)!}\left(\frac{d}{dy}\right)^{2n}\frac{1}{2\cos(2y)-1}\Bigg|_{y=0} \tag{44}$$

$$\left.\begin{matrix}\zeta(2n+1,\frac{1}{3})\\ \\ \zeta(2n+1,\frac{2}{3})\end{matrix}\right\} = \frac{1}{2}(3^{2n+1}-1)\zeta(2n+1)$$
$$\pm\frac{\pi^{2n+1}\sqrt{3}}{2(2n)!}\left(\frac{d}{dy}\right)^{2n}\frac{1}{2\cos(2y)+1}\Bigg|_{y=0} \tag{45}$$

In particular, setting $n = 1$ and $n = 3$ in Theorems 3.1 and 3.3, we obtain

$$\begin{array}{ll}\zeta(3,\frac{1}{4}) = \pi^3 + 28\zeta(3) & \zeta(3,\frac{1}{3}) = \dfrac{2\pi^3}{3\sqrt{3}} + 13\zeta(3)\\ \zeta(3,\frac{1}{6}) = 2\sqrt{3}\pi^3 + 91\zeta(3) & \zeta(5,\frac{1}{4}) = \dfrac{5\pi^5}{3} + 496\zeta(5)\\ \zeta(5,\frac{1}{3}) = \dfrac{2\pi^5}{3\sqrt{3}} + 121\zeta(5) & \zeta(5,\frac{1}{6}) = \dfrac{22\pi^5}{\sqrt{3}} + 3751\zeta(5)\end{array}$$

In [9] (see also [6], p. 79), Glaisher defined special numbers H_n and G_n analogous to the Eulerian numbers

$$\frac{3}{2(2\cos(x)-1)} = \sum_{k=0}^{\infty} H_n \frac{x^n}{n!} \tag{46}$$

$$\frac{3}{2(2\cos(x)+1)} = \sum_{k=0}^{\infty} \frac{G_n}{n+1}\frac{x^n}{n!} \tag{47}$$

Here are the first few values

$$\begin{aligned} &H_0 = \frac{3}{2}, \quad H_2 = 3, \quad H_4 = 33 \\ &H_6 = 903,\ H_8 = 46113,\ H_{2n+1} = 0 \end{aligned}$$

and

$$\begin{aligned} &G_0 = \frac{1}{2}, \quad G_2 = 1, \quad G_4 = 5 \\ &G_6 = 49,\ G_8 = 809,\ G_{2n+1} = 0 \end{aligned}$$

It is easy to see that representations (44) and (45) can be rewritten in terms of the Glaisher constants.

Corollary 3.4 *For* $n \in \mathrm{N}$,

$$\left.\begin{array}{l} \zeta(2n+1,\frac{1}{6}) \\ \\ \zeta(2n+1,\frac{5}{6}) \end{array}\right\} = \frac{1}{2}(3^{2n+1}-1)(2^{2n+1}-1)\zeta(2n+1) \pm \frac{(2\pi)^{2n+1}}{2\sqrt{3}(2n)!} H_{2n}$$

$$\left.\begin{array}{l} \zeta(2n+1,\frac{1}{3}) \\ \\ \zeta(2n+1,\frac{2}{3}) \end{array}\right\} = \frac{1}{2}(3^{2n+1}-1)\zeta(2n+1) \pm \frac{(2\pi)^{2n+1}}{2\sqrt{3}(2n+1)!} G_{2n}$$

On the other hand, inverting Corollary 3.4, Glaisher's numbers are just a combination of the Hurwitz functions:

$$H_{2n} = \frac{\sqrt{3}\,(2n)!}{(2\pi)^{2n+1}} \left(\zeta(2n+1,\frac{1}{6}) - \zeta(2n+1,\frac{5}{6})\right), \quad n \in \mathrm{N} \tag{48}$$

$$G_{2n} = \frac{\sqrt{3}\,(2n+1)!}{(2\pi)^{2n+1}} \left(\zeta(2n+1,\frac{1}{3}) - \zeta(2n+1,\frac{2}{3})\right), \quad n \in \mathrm{N} \tag{49}$$

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References

[1] V. S. Adamchik, Multiple Gamma function and its application to computation of series, *The Ramanujan Journal* **9**(2005), 271-288.

[2] V. S. Adamchik, Symbolic and numeric computations of the Barnes function, *Comp. Phys. Comm.* **157**(2004), 181-190.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[4] H. Bateman, A. Erdelyi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, Toronto, London, 1953.

[5] J. Choi, H. M. Srivastava, V. S. Adamchik, Multiple Gamma and related functions, *Appl. Math. and Comp.* **134** (2003), 515–533.

[6] A. Fletcher, J. C. P. Miller, L. Rosenhead, L. J. Comrie, *An Index of Mathematical Tables*, Vol. 1, 2nd ed., Blackwell, Oxford and Addison-Wesley, Reading, MA, 1962.

[7] J. W. L. Glaisher, On the product $1^1, 2^2, 3^3, \ldots, n^n$, *Messenger of Math.* **7**(1887), 43-47.

[8] J. W. L. Glaisher, On certain numerical products, *Messenger of Math.* **23**(1893), 145-175.

[9] J. W. L. Glaisher, On a set of coefficients analogous to the Eulerian numbers, *Proc. London Math. Soc.* **31**(1899), 216-235.

[10] M. E. Hoffman, Derivative polynomials for tangent and secant, *American Math. Monthly* **102**(1995), 23-30.

[11] Kinkelin, Ueber eine mit der Gammafunction verwandte Transcendente und deren Anwendung auf die Integralrechnung, *J. Reine Angew. Math.* **57**(1860), 122–158.

[12] P. M. Knopf, The operator $(z\,d/dz)^n$ and its applications to series, *Math. Magazine* **76**(2003), 364-371.

[13] D. Knuth, T. Buckholtz, Computation of tangent, Euler, and Bernoulli numbers, *Math. Comp.* **21**(1967), 663-688.

[14] K. S. Kolbig, The polygamma function and the derivatives of the cotangent function for rational arguments, *CERN-IT-Reports, CERN-CN-96-005*, 1996.

[15] W. Magnus, F. Oberhettinger, R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.

[16] K. N. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, New York, 2000.

[17] H. M. Srivastava, J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Series on Mathematics and Its Applications, vol. 531, Kluwer Academic Publishers, Dordrecht, 2001.