The evaluation of integrals of Bessel functions via G-function identities

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Abstract

A few transformations are presented for reducing certain cases of Meijer's G-function to a G-function of lower order. Their applications to the integration of a product of Bessel functions are given. The algorithm has been implemented within *Mathematica* 3.0.

1 Introduction

In this note we continue to discuss the algorithm of obtaining analytical solutions to definite integals by using the method of the Mellin integral transform. The overall idea of this method has been given in Adamchik and Marichev [2]. It was shown there that by applying the Mellin integral transform to an improper integral, the latter can be represented by means of Meijer's G-function, which is, in other words, a Mellin-Barnes contour integral in the complex plane. The success of this method (at this stage) depends on two things: first, the Mellin image of a correspondent function must exist and, second, the Mellin image should be represented in terms of gamma functions. If these conditions are satisfied, then the next question, which arises immediately, is how to evaluate the contour integral. Speaking generically, it can be evaluated as a sum of residues under some limitations. In the special functions literature this fact is known as Slater's theorem (see [5]). According to this theorem the Mellin-Barnes contour integral can almost always be expressed in terms of hypergeometric functions. There is a special case of the G-function where that is not so. This is called the singular case or logarithmic case.

The article Adamchik and Kolbig [1] considers a singular case of the G-function arising from the integral of a product of two polylogarithms. It was shown that there are no serious obstacles to evaluating a G-function in the special case algorithmically. Although *Mathematica* is doing this already, there is a design problem here. It is caused by a bulky form of the G-function representation which involves finite and infinite sums with psi (polygamma) functions in the summands in addition to hypergeometric functions. The most significant problem, then, is how to eliminate logarithmic cases of Meijer's G-function. Here we give some useful transformations that allow us to reduce a G-function to another one of lower order. Such transformations are especially useful in logarithmic cases of Meijer's G-function.

2 Reduction Formulas

To simplify the exposition, it is convenient to introduce the following notations:

$$(a)_n = \prod_{i=1}^{n} (a + i - 1),$$

$$\Gamma(\vec{a}_{n,p} + s) = \prod_{i=n}^{p} \Gamma(a_i + s).$$

Throughout this section $z \neq 0$ and m, n, p and q are integers with

$$q \geq 1, \quad 0 \leq n \leq p \leq q, \quad 0 \leq m \leq q.$$

and

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \frac{1}{2\pi i} \oint_L \frac{\Gamma(\vec{b}_{1,m} + s)\,\Gamma(1 - \vec{a}_{1,n} - s)}{\Gamma(\vec{a}_{n+1,p} + s)\,\Gamma(1 - \vec{b}_{m+1,q} - s)}\, z^{-s} ds$$

where the contour of the integration L is a left loop, beginning and ending at $-\infty$ and encircling all poles of $\Gamma(b_k+s)$, $k = 1, 2, \ldots, m$ in the positive direction, but none of the right series of the poles of $\Gamma(1 - a_k - s)$, $k = 1, 2, \ldots, n$.

Proposition 1 *Suppose $a-b$ is an integer and a_k-b, $k = 1, \ldots, n$ are not positive integers. Then*

CASE 1. $a - b$ is positive

$$\begin{aligned} G_{p+1,q+2}^{m+1,n+1}\left(z \,\middle|\, \begin{matrix} a, a_p \\ a, b_q, b \end{matrix}\right) &= (-1)^{a-b}\; G_{p,q+1}^{m+1,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b, b_q \end{matrix}\right) - \\ &(-1)^{a-b} \sum_{k=1}^{a-b} \operatorname*{Res}_{s=-a+k} \left[\frac{\Gamma(b+s)\,\Gamma(\vec{b}_{1,m} + s)\,\Gamma(1 - \vec{a}_{1,n} - s)}{\Gamma(\vec{a}_{n+1,p} + s)\,\Gamma(1 - \vec{b}_{m+1,q} - s)}\, z^{-s} \right] \end{aligned} \qquad (1)$$

CASE 2. $a - b$ *is negative or zero*

$$G^{m+1,n+1}_{p+1,\,q+2}\left(z\,\middle|\,\begin{matrix} a, a_p \\ a, b_q, b \end{matrix}\right) = (-1)^{b-a}\;\; G^{m+1,n}_{p,\,q+1}\left(z\,\middle|\,\begin{matrix} a_p \\ b, b_q \end{matrix}\right) \tag{2}$$

Proof.

The proof could be performed in two steps. First we find an algebraic transformation which reduces the order of the G-function and then we determine limitations under which that transformation becomes correct.

Consider the integrand of the G-function on the left side of equations (1) and (2). For the purpose of clear exposition, we will concentrate on those gamma functions in the integrand that involve parameters a and b:

$$\frac{\Gamma(a+s)\,\Gamma(1-a-s)}{\Gamma(1-b-s)}$$

Applying a reflection formula for the gamma function

$$\Gamma(1-a-s) = \frac{\pi}{\Gamma(a+s)\,\sin\pi(a+s)},$$

the above expression can be rewritten as

$$\Gamma(b+s)\,\frac{\sin\pi(b+s)}{\sin\pi(a+s)}$$

Given that $a - b$ is an integer, the quotient of sine functions simplifies and finally we obtain

$$\frac{\Gamma(a+s)\,\Gamma(1-a-s)}{\Gamma(1-b-s)} = (-1)^{a-b}\,\Gamma(b+s)$$

In terms of G-functions this means that we can transform the G-function from the left side of the equation (1) to another one of lower order:

$$G^{m+1,n+1}_{p+1,\,q+2}\left(z\,\middle|\,\begin{matrix} a, a_p \\ a, b_q, b \end{matrix}\right) \Rightarrow (-1)^{a-b}\;\; G^{m+1,n}_{p,\,q+1}\left(z\,\middle|\,\begin{matrix} a_p \\ b, b_q \end{matrix}\right) \tag{3}$$

Now we have to figure out how this algebraic transformation has changed the poles of the integrand. The below picture shows that in the case where $a - b$ is a positive integer, the transformation has added a finite number of new poles at $-a+1, -a+2, \ldots, -a+(a-b)$:

$$\begin{array}{ccccccc} \dots & \star & \star & \star & \star & \dots & \star \\ \dots & -a-1 & -a & \multicolumn{4}{c}{\underbrace{-a+1 \qquad -a+2 \qquad \dots \qquad -b=-a+(a-b)}} \end{array}$$

It follows that, to make the relation (3) completely correct we need to substract the sum of residues at the new poles from the G-function on the right side of (3). To complete this case, we need to check if the contour of integration separates these new poles from the right series of poles. This is provided by the restriction

$$-b - l \neq 1 - a_k + j, \; k = 1, \dots, n,$$

where l and j are nonnegative integers.

In the second case, where $b-a$ is a positive integer, the above transformation (3) not change any left side poles of the G-function and consequently equation (2) is correct.

Proposition 2 *Suppose* $a-b$ *is an integer. Then*

$$G^{m,\, n+1}_{p+2,q+1}\left(z \,\middle|\, \begin{matrix} a, a_p, b \\ b_q, b \end{matrix}\right) = (-1)^{a-b} \; G^{m,\, n}_{p+1,q}\left(z \,\middle|\, \begin{matrix} a_p, a \\ b_q \end{matrix}\right) \tag{4}$$

Proof. In very much the same way as it is done for the previous propositions, we can prove this one. The only difference is that in this case the transformation by using a reflection formula is invariant with respect to those poles of the integrand which belong to the domain encircled by the contour L.

Proposition 3 *Suppose* $a-b$ *and* $c-a+1$ *are positive integers and* $a_k - b$, $k = 1, \ldots, n$ *are not positive integers. Then*

$$\begin{aligned} G^{m+1,n+1}_{p+1,\,q+2}\left(z \,\middle|\, \begin{matrix} a, a_p \\ c, b_q, b \end{matrix}\right) &= (-1)^{a-b}\left(a - z\frac{d}{dz}\right)_{c-a} G^{m+1,n}_{p,\,q+1}\left(z \,\middle|\, \begin{matrix} a_p \\ b, b_q \end{matrix}\right) - \\ &(-1)^{a-b}\sum_{k=1}^{c-b} \operatorname*{Res}_{s=1-b-k}\left[\frac{\Gamma(\vec{b}_{1,m}+s)\,\Gamma(b+s)\,\Gamma(1-\vec{a}_{1,n}-s)}{\Gamma(\vec{a}_{n+1,p}+s)\,\Gamma(1-\vec{b}_{m+1,q}-s)}\,(a+s)_{c-a}\,z^{-s}\right] \end{aligned} \tag{5}$$

Proof. Let us proceed from this part of the integrand

$$\frac{\Gamma(c+s)\,\Gamma(1-a-s)}{\Gamma(1-b-s)} \tag{6}$$

According to the above conditions, suppose

$$a-b=r \quad \text{and} \quad c-a+1=k$$

where r and k are positive integers.

Then

$$\Gamma(1-b-s) = \Gamma(1-a-s+r) = (1-a-s)_r\,\Gamma(1-a-s) = (-1)^r\,(a-r+s)_r\,\Gamma(1-a-s)$$

and

$$\Gamma(c+s) = \Gamma(k+a-1+s) = \Gamma((a-r+s)+r+k-1) = (a-r+s)_{r+k-1}\,\Gamma(a-r+s)$$

Hence, the expression (6) can be rewritten as follows

$$\frac{\Gamma(c+s)\,\Gamma(1-a-s)}{\Gamma(1-b-s)} = (-1)^r\,\frac{(a-r+s)_{r+k-1}}{(a-r+s)_r}\,\Gamma(a-r+s) = (-1)^{a-b}(a+s)_{c-a}\Gamma(b+s)$$

and in terms of integrals we have

$$G^{m+1,n+1}_{p+1,\,q+2}\left(z \,\middle|\, \begin{matrix} a, a_p \\ c, b_q, b \end{matrix}\right) \Rightarrow$$

$$\frac{(-1)^{a-b}}{2\pi i}\oint_L \frac{\Gamma(b+s)\,\Gamma(\vec{b}_{1,m}+s)\,\Gamma(1-\vec{a}_{1,n}-s)}{\Gamma(\vec{a}_{n+1,p}+s)\,\Gamma(1-\vec{b}_{m+1,q}-s)}\,(a+s)_{c-a}\,z^{-s}ds$$

Represent the inner polynomial $(a+s)_{c-a}$ with respect to s through derivatives with respect to z. In operator form this is

$$s^k\, z^{-s} = (-z\,\frac{d}{dz})^k\, z^{-s}$$

or

$$(a+s)_{c-a} z^{-s} = (a - z\frac{d}{dz})_{c-a}\, z^{-s}$$

Hence,

$$G^{m+1,n+1}_{p+1,\,q+2}\left(z\,\middle|\,\begin{matrix} a, a_p \\ c, b_q, b\end{matrix}\right) \Rightarrow (-1)^{a-b}(a - z\,\frac{d}{dz})_{c-a}\;\; G^{m+1,n}_{p,\,q+1}\left(z\,\middle|\,\begin{matrix} a_p \\ b, b_q\end{matrix}\right) \tag{7}$$

Again we should figure out how this transformation has changed the poles of the integrand. It is clear that formally the parameter c from the G-function on the left side has been replaced by b. This means that the transformation has added new poles to the integrand at points $-c+1, \ldots, -b$. Consequently, we have to subtract the sum of residues at these points from the G-function on the right side (7) and we immediately accomplish the proposition.

3 Examples

In order to show how and where these propositions are working, we consider a few improper integrals of a product of Bessel functions.

3.1 Example 1

In the article McPhedrane t al. [4] the authors investigated the following class of integrals

$$\int_0^\infty x\,\exp(-x^2/z)\,J_n(x)\,Y_n(x)\,dx$$

and discovered common analytical formulas for their solutions. In this section we show how to evaluate this type of integral algorithmically. Consider the most interesting case, when the parameter n is an integer. Without loss of generality we can set the parameter n to, for example, 2. In the first step we use the Mellin integral transform to represent the integral as Meijer's G-function:

$$\int_0^\infty x \exp(-x^2/z)\, \mathrm{J}_2(x)\, \mathrm{Y}_2(x)\, dx = \frac{z}{2\sqrt{\pi}} \; \mathrm{G}^{2,2}_{3,4}\left(z \,\middle|\, \begin{matrix} 0, 1/2, -1/2 \\ 0, 2, -1/2, -2 \end{matrix} \right)$$

According to the proposition 1 (with $a = 0$ and $b = -2$) the order of this G-function can be reduced. We have

$$\mathrm{G}^{2,2}_{3,4}\left(z \,\middle|\, \begin{matrix} 1/2, 0, -1/2 \\ 2, 0, -1/2, -2 \end{matrix} \right) = \mathrm{G}^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} 1/2, -1/2 \\ 2, -2, -1/2 \end{matrix} \right) -$$
$$\operatorname*{Res}_{s=1} \left[\frac{\Gamma(2+s)\,\Gamma(-2+s)\,\Gamma(1/2-s)}{\Gamma(-1/2+s)\,\Gamma(3/2-s)}\, z^{-s} \right] -$$
$$\operatorname*{Res}_{s=2} \left[\frac{\Gamma(2+s)\,\Gamma(-2+s)\,\Gamma(1/2-s)}{\Gamma(-1/2+s)\,\Gamma(3/2-s)}\, z^{-s} \right]$$

Again, we can reduce the order of the G-function here, by applying the proposition 2, where $a = 1/2$ and $b = -1/2$. It implies

$$\mathrm{G}^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} 1/2, -1/2 \\ 2, -2, -1/2 \end{matrix} \right) = \mathrm{G}^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} 1/2 \\ 2, -2 \end{matrix} \right)$$

The G-function on the right side is a known integral representation for the modified Bessel function $\mathrm{K}_\nu(x)$ (see [3, p.307]):

$$\mathrm{G}^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} c \\ a, b \end{matrix} \right) = \frac{z^{c-1/2}\, \mathrm{K}_{a-c+1/2}(z/2)}{\exp(z/2)\sqrt{\pi}} \tag{8}$$

where $2\,c - a - b - 1 = 0$.

Finally, we obtain

$$\int_0^\infty x \exp(-x^2/z)\, \mathrm{J}_2(x)\, \mathrm{Y}_2(x)\, dx = \frac{-2}{\pi} + \frac{4}{z\,\pi} - \frac{z\, \mathrm{K}_2(z/2)}{2\,\pi \exp(z/2)} \tag{9}$$

3.2 *Example 2*

As a similar example, consider the integral:

$$\int_0^\infty x \exp(-x^2/z)\, \mathrm{I}_3(x)\, \mathrm{K}_3(x)\, dx$$

With the same technique we can express this as a G-function:

$$\int_0^\infty x \exp(-x^2/z)\, \mathrm{I}_3(x)\, \mathrm{K}_3(x)\, dx = \frac{z}{4\sqrt{\pi}} \; G_{2,3}^{2,2}\left(z \,\middle|\, \begin{matrix} 1/2, 0 \\ 0, 3, -3 \end{matrix}\right)$$

For $a = 0$ and $b = -3$ it follows from the proposition 1 that the above G-function reduce to this one:

$$G_{2,3}^{2,2}\left(z \,\middle|\, \begin{matrix} 1/2, 0 \\ 0, 3, -3 \end{matrix}\right) = - \; G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1/2 \\ 3, -3 \end{matrix}\right) +$$

$$\operatorname*{Res}_{s=1} \left[\Gamma(3+s)\,\Gamma(-3+s)\,\Gamma(1/2-s)\, z^{-s}\right] +$$

$$\operatorname*{Res}_{s=2} \left[\Gamma(3+s)\,\Gamma(-3+s)\,\Gamma(1/2-s)\, z^{-s}\right] +$$

$$\operatorname*{Res}_{s=3} \left[\Gamma(3+s)\,\Gamma(-3+s)\,\Gamma(1/2-s)\, z^{-s}\right]$$

Evaluating residues and using this representation of the Bessel function $\mathrm{K}_\nu(x)$ from Luke (see [3, p.307]):

$$G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} c \\ a, b \end{matrix}\right) = \frac{z^{c-1/2}\, \mathrm{K}_{a-c+1/2}(z/2)\, \exp(z/2)\sqrt{\pi}}{\cos \pi\,(a-c+1/2)} \tag{10}$$

where $2\,c - a - b - 1 = 0$, we arrive at the result

$$\int_0^\infty x \exp(-x^2/z)\, \mathrm{I}_3(x)\, \mathrm{K}_3(x)\, dx = \frac{-(32 + 16\,z + 3\,z^2)}{2\,z^2} + \frac{\exp(z/2)\, z\, \mathrm{K}_3(z/2)}{4} \tag{11}$$

3.3 Example 3

Consider another example from the article McPhedran et al. [4]:

$$\int_0^\infty x^3 \exp(-x^2/z)\, \mathrm{J}_2(x)\, \mathrm{Y}_2(x)\, dx$$

This can be written as a G-function as follows:

$$\frac{z^2}{2\sqrt{\pi}} \; G_{3,4}^{2,2}\left(z \,\middle|\, \begin{matrix} 1/2, -1, -1/2 \\ 2, 0, -1/2, -2 \end{matrix}\right)$$

Applying the proposition 2 with $a = 1/2$ and $b = -1/2$ we reduce this G-function to

$$\frac{-z^2}{2\sqrt{\pi}} \; G^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} -1, 1/2 \\ 2, 0, -2 \end{matrix}\right)$$

Now we can easily observe that the proposition 3 is applicable to this function. With $a = -1,\ b = -2$ and $c = 0$ it follows that

$$G^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} -1, 1/2 \\ 2, 0, -2 \end{matrix}\right) = G^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} 1/2 \\ 2, -2 \end{matrix}\right) + z\frac{d}{dz}\, G^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} 1/2 \\ 2, -2 \end{matrix}\right) +$$
$$\operatorname*{Res}_{s=1}\left[\frac{\Gamma(2+s)\,\Gamma(-2+s)\,(s-1)}{\Gamma(1/2+s)\, z^S}\right] + \operatorname*{Res}_{s=2}\left[\frac{\Gamma(2+s)\,\Gamma(-2+s)\,(s-1)}{\Gamma(1/2+s)\, z^S}\right]$$

Applying the formula (8) and evaluating residues, we obtain

$$\int_0^\infty x^3 \exp(-x^2/z)\, J_2(x)\, Y_2(x)\, dx = \tag{12}$$
$$\frac{-4}{\pi} + \frac{z^2\,(2+z)\, K_0(z/2)}{4\,\pi \exp(z/2)} + \frac{z\,(8+4z+z^2)\, K_1(z/2)}{4\,\pi \exp(z/2)}$$

3.4 *Example 4.*

Consider the integral with modified Bessel functions

$$\int_0^\infty x^5 \exp(-x^2/z)\, I_3(x)\, K_3(x)\, dx$$

which is the folowing G-function

$$\frac{z^3}{4\sqrt{\pi}} \; G^{2,2}_{2,3}\left(z \,\middle|\, \begin{matrix} 1/2, -2 \\ 3, 0, -3 \end{matrix}\right)$$

According to the proposition 3 with $a = -2$, $b = -3$ and $c = 0$, we have

$$G^{2,2}_{2,3}\left(z\,\middle|\,\begin{matrix}1/2,-2\\3,0,-3\end{matrix}\right) = -2\ G^{2,1}_{1,2}\left(z\,\middle|\,\begin{matrix}1/2\\3,-3\end{matrix}\right) - 3\,z\,\frac{d}{dz}\ G^{2,1}_{1,2}\left(z\,\middle|\,\begin{matrix}1/2\\3,-3\end{matrix}\right) -$$

$$(-z\,\frac{d}{dz})^2\ G^{2,1}_{1,2}\left(z\,\middle|\,\begin{matrix}1/2\\3,-3\end{matrix}\right) +$$

$$\sum_{k=1}^{3}\operatorname*{Res}_{s=k}\left[\Gamma(3+s)\,\Gamma(-3+s)\,\Gamma(1/2-s)\,(s-1)\,(s-2)\,z^{-s}\right]$$

With the representation (10), this implies

$$\int_0^\infty x^5\ \exp(-x^2/z)\,I_3(x)\,K_3(x)\,dx = -32 +$$
$$\frac{1}{8}\exp(z/2)\,z^2\,(32-16\,z+5\,z^2-z^3)\,K_0(z/2) + \tag{13}$$
$$\frac{1}{8}\exp(z/2)\,z\,(128-64\,z+24\,z^2-6\,z^3+z^4)\,K_1(z/2)$$

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