THE ALGORITHM FOR CALCULATING INTEGRALS OF HYPERGEOMETRIC
TYPE FUNCTIONS AND ITS REALIZATION IN REDUCE SYSTEM

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The most voluminous bibliography of the analytical methods for
calculating of integrals is represented in the article [19]. It is
shown there that the most effective and the simpliest algorithm of
analytical integration was made by O.I.Marichev [8, 9, 12]. Later
it was realised in the reference-books [16-18, 20]. This algorithm
allows us to calculate definite and indefinite integrals of the
products of elementary and special functions of hypergeometric type.
It embraces about 70 per cent of integrals which are included in the
world reference-literature. It allows to calculate many other integ-
rals too.

The present article contains short description of this algori-
thm and its realization in the REDUCE system during the process of
creation of INTEGRATOR system. Only one general method of integrati-
on is known to be realized on the computers, i.e. criterion algo-
rithm for calculating of indefinite integrals of elementary functi-
ons through elementary functions by themselves (the authors of it
are M. Bronstein and other).

The idea of our algorithm is in the following. The initial in-
tegrals is transformed to contour integral from the ratio of products
of gamma-functions by means of Mel-
lin transform and parseval equali-
ty. The residue theorem is used for
the calculating of the received

integral which due to the strict rules results in sums of hypergeo-
metric series. The value of integral itself and the integrand func-
tions are the special cases of the well-known Meijer's G-function
[4, 7, 8, 12, 14, 18].

Programming packet is realized in programming languages PASCAL
and REDUCE. It also offers the opportunity of finding the values for
some classical integral transforms (Laplace, Hankel,Fourier, Mellin
and etc.). The REDUCE's part of packet contains the main properties
of the well-known special functions, such as the Bessel and gamma-
functions and kindred functions, Anger function, Weber function,
Whittaker functions, generalized hypergeometric functions. Special
place in the packet is occupied by Meijers's G-function for which
the main properties such as finding the particular cases and repre-
sentation by means of hypergeometric series are realized.

1. Preliminary information. As it is known from the residue
theory the following formula holds

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{A}(s)ds = \sum_{k \in \mathbb{K}} \operatorname*{res}_{s=s_k} \mathcal{A}(s),  \tag{1}$$

where $\mathcal{L}$ is the closed contour, embracing the poles $s = a_k$, $k \in \mathbb{K}$,
of the analytical function $\mathcal{A}(s)$ in positive direction. This formula
is true also for infinite contours (for example, if $\mathcal{L}=(\gamma-i\infty, \gamma+i\infty)$)
under the additional condition that the integral converges when we
carry out of a limit transition from closed contour to discontinuo-
us infinite ones.

Let $\mathcal{A}(s)=\Gamma(a+bs)\varphi(s)$, where $\varphi(s)$ is analytical function in-
side the contour $\mathcal{L}$ and $\Gamma(s)$ is the gamma-function. Then

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(a+bs)\varphi(s)ds = \sum_{k \in \mathbb{K}} \frac{(-1)^k}{bk!}\varphi\left(-\frac{a+k}{b}\right),  \tag{2}$$

where $\mathbb{K} \in \{0, 1, 2,...\}$ with condition that the poles $s=-(a+k)/b$ of
the function $\Gamma(a+bs)$ with the numbers $k \in \mathbb{K}$ are placed inside the con-
tour $\mathcal{L}$, embracing in positive direction. For example, the formula
(2) gives the interesting consequences for the power function $(1+x)^\alpha$.

Indeed, the following distribution takes place

$$(1+x)^{\alpha} = \begin{cases} \sum_{k=0}^{\infty} (-\alpha)_k \; x^k \; \dfrac{(-1)^k}{k!} & , \quad |x|<1 , \\ x^{\alpha} \sum_{k=0}^{\infty} (-\alpha)_k \; x^{-k} \; \dfrac{(-1)^k}{k!} & , \quad |x|>1 , \end{cases} \qquad (3)$$

where $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol. If we choose in the formula (2) at first $a=0$, $b=1$, $\varphi(s)=x^{-s} \, \Gamma(-\alpha-s)/\Gamma(\alpha)$, at second $a=-\alpha$, $b=-1$, $\varphi(s)=x^{-s} \, \Gamma(s)/\Gamma(-\alpha)$ and take $\mathbb{K}=\{0, 1, 2, \ldots\}$ then with account of (3) we shall receive the equalities

$$(1+x)^{\alpha} = \left\{ \begin{array}{ll} \sum_{k=0}^{\infty} \operatorname*{res}_{s=-k} \left[ \dfrac{\Gamma(s)\Gamma(-\alpha-s)}{\Gamma(-\alpha)} \, x^{-s} \right] & , \quad |x|<1 \\ -\sum_{k=0}^{\infty} \operatorname*{res}_{s=k-\alpha} \left[ \dfrac{\Gamma(s)\Gamma(-\alpha-s)}{\Gamma(-\alpha)} \, x^{-s} \right] & , \quad |x|>1 \end{array} \right\} = \qquad (4)$$

$$= \frac{1}{2\pi i} \int_{\mathscr{L}} \frac{\Gamma(s)\Gamma(-\alpha-s)}{\Gamma(-\alpha)} \, x^{-s} \; ds.$$

The contour $\mathscr{L}$ here must embrace all poles of the type $s=-k$ (for $|x|<1$) or the kind $s=k-\alpha$ (for $|x|>1$) passing to left (or right) infinity and what's more it may be straight line $\mathscr{L}=(\gamma-i\infty, \gamma+i\infty)$, $0<\gamma<-\operatorname{Re}\alpha$. These restrictions and condition $|\arg x|<\pi$ are sufficient for convergence of the corresponding integrals by contour $\mathscr{L}$ and power series.

This example demonstrates the process of transform from functions presented through power series with products of the Pochhammer symbols to integrals from ratios of products of gamma-functions. Such series are called hypergeometric series and such integrals are called Mellin-Barnes type integrals. Highly general class of such type integrals is designated by the symbol $G_{pq}^{mn}(z)$ which is defined by the formula

$$G_{pq}^{mn}(z) = G_{pq}^{mn}\left[ z \; \middle| \; \begin{array}{c} (a_p) \\ (b_q) \end{array} \right] = \frac{1}{2\pi i} \int_{\mathscr{L}} g(s) \; z^{-s} \; ds, \qquad (5)$$

where

$$g(s) = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j+s) \prod\limits_{j=1}^{n} \Gamma(1-\alpha_j-s)}{\prod\limits_{j=1+n}^{p} \Gamma(a_j+s) \prod\limits_{j=1+m}^{q} \Gamma(1-b_j-s)} \,, \qquad 0 \leq m \leq q, \; 0 \leq n \leq p. \qquad (6)$$

The function (5) was introduced and studied by C.S.Meijer in number of the articles of 1936-1956 years (see [4]) and later it was called Meijer's G-function. In its definition it is supposed that the contour $\mathcal{L}$ may be of three kinds: $\mathcal{L}_{-\infty}$, $\mathcal{L}_{+\infty}$ or $\mathcal{L}_{i\infty}$. The contour $\mathcal{L}_{-\infty}$ (accordingly $\mathcal{L}_{+\infty}$) is the left (or right) loop which is arranged in some horizontal belt, begins in the point of $-\infty+i\lambda$ ($+\infty+i\lambda$), leaves all poles $s=-b_j-\ell$, $j=1, 2, \ldots, m$, $\ell=0, 1, 2, \ldots$, of the integrand function at the left and all poles $s=1-a_j+\ell$, $j=1, 2, \ldots, n$, $\ell=0, 1, 2, \ldots$, at the right of the contour and at last contour $\mathcal{L}$ finishes in the point $-\infty+i\delta$ ($+\infty+i\delta$), where $\lambda < \delta$. The contour $\mathcal{L}_{i\infty}$ begins in the point $\gamma-i\infty$ and finishes in the point $\gamma+i\infty$. It separates these poles in the same way as $\mathcal{L}_{\pm\infty}$. In particular if the condition of separation of the poles allows the contour $\mathcal{L}$ to be rectilinear one then it may be considered to be a straight line ($\gamma-i\infty$, $\gamma+i\infty$).

The integral (5),(6) converges if any of the following four conditions are fulfiled:

1) $\mathcal{L}=\mathcal{L}_{i\infty}$, $c^*>0$, $|\arg z|<c^*\pi$;
2) $\mathcal{L}=\mathcal{L}_{i\infty}$, $c^*\geq0$, $|\arg z|=c^*\pi$, $(q-p)\gamma<-\operatorname{Re}\mu$;
3) $\mathcal{L}=\mathcal{L}_{-\infty}$; $p<q$, $0<|z|<\infty$ or $p=q$, $0<|z|<1$ or $p=q$, $c^*\geq0$, $|z|=1$, $\operatorname{Re}\mu<0$;
4) $\mathcal{L}=\mathcal{L}_{+\infty}$; $p>q$, $0<|z|<\infty$ or $p=q$, $|z|>1$ or $p=q$, $c^*\geq0$, $|z|=1$, $\operatorname{Re}\mu<0$;

Here we denoted

$$c^*=m+n-(p+q)/2, \quad \mu=\sum_{j=1}^{p} b_j - \sum_{j=1}^{q} a_j + (p-q)/2 + 1, \quad \gamma=\lim_{\substack{s\to\infty \\ s\in\mathcal{L}_{i\infty}}} \operatorname{Re} s. \quad (7)$$

If $c^*>0$ then G-function is analytical function for $|\arg z|<c^*\pi$. If $q=p$ then G-function, generally spearing, is piece-analytical function and what's more if $c^*>0$ then its components continue one another analytically through the circle $|z|=1$ in the sector $|\arg z|<c^*\pi$. If $c^*=0$, $q=p$, $\operatorname{Re}\mu<0$, then these components are continued for $z=1$ along the ray $\arg z=0$.

Mellin-Barnes integrals, i.e. Meijer's G-functions (5),(6) produce the full analytical functions. Any finite linear combinations of these functions we shall call hypergeometric type functions. The fullest table of participal cases of G-functions containing about

1000 formulae are given in the reference-book [18]. In particular the following representations are true:

$$H(1-x) = \left\{ \begin{matrix} 1, & 0<x<1 \\ 0, & x>1 \end{matrix} \right\} = G_{11}^{10}\left[x \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right], \tag{8}$$

$$(1-x)_+^{\alpha-1} = \left\{ \begin{matrix} (1-x)^{\alpha-1}, & 0<x<1 \\ 0, & x>1 \end{matrix} \right\} = \Gamma(\alpha)G_{11}^{10}\left[x \,\middle|\, \begin{matrix} \alpha \\ 0 \end{matrix}\right]. \tag{9}$$

$$(1+x)^{\alpha} = \frac{1}{\Gamma(-\alpha)} G_{11}^{11}\left[x \,\middle|\, \begin{matrix} \alpha+1 \\ 0 \end{matrix}\right], \quad (1-x)^{-1} = \pi\, G_{22}^{11}\left[x \,\middle|\, \begin{matrix} 0, 1/2 \\ 0, 1/2 \end{matrix}\right], \tag{10}$$

$$e^{-x} = G_{01}^{10}\left[x \,\middle|\, \begin{matrix} \cdot \\ 0 \end{matrix}\right], \qquad \ln(1+x) = G_{22}^{12}\left[x \,\middle|\, \begin{matrix} 1, 1 \\ 1, 0 \end{matrix}\right], \tag{11}$$

$$\arctan(x) = \frac{1}{2} G_{22}^{12}\left[x^2 \,\middle|\, \begin{matrix} 1/2, 1 \\ 1/2, 0 \end{matrix}\right], \quad \operatorname{erf}(x) = \pi^{-1/2} G_{12}^{11}\left[x^2 \,\middle|\, \begin{matrix} 1 \\ 1/2, 0 \end{matrix}\right]. \tag{12}$$

$$\left\{ \begin{matrix} \sin(x) \\ \cos(x) \end{matrix} \right\} = \sqrt{\pi}\, G_{02}^{10}\left[x^2/4 \,\middle|\, \begin{matrix} \cdot \\ (1\pm1)/4, (1\mp1)/4 \end{matrix}\right], \tag{13}$$

$$J_\nu(x) = G_{02}^{10}\left[x^2/4 \,\middle|\, \begin{matrix} \cdot \\ \nu/2, -\nu/2 \end{matrix}\right], \quad K_\nu(x) = \frac{1}{2} G_{02}^{20}\left[x^2/4 \,\middle|\, \begin{matrix} \cdot \\ \nu/2, -\nu/2 \end{matrix}\right]. \tag{14}$$

$$I_\mu(x)K_\nu(x) = \frac{1}{2} \pi^{-1/2} G_{24}^{22}\left[x^2 \,\middle|\, \begin{matrix} 0, & 1/2 \\ (\mu+\nu)/2, (\mu-\nu)/2, (\nu-\mu)/2, -(\mu+\nu)/2 \end{matrix}\right]. \tag{15}$$

$$P_n(x)H(1-x) = G_{22}^{20}\left[x^2 \,\middle|\, \begin{matrix} (1-n)/2, & 1+n/2 \\ 0, & 1/2 \end{matrix}\right], \tag{16}$$

$${}_2F_1(a, b; c; -x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G_{22}^{12}\left[x \,\middle|\, \begin{matrix} 1-a, 1-b \\ 0, 1-c \end{matrix}\right]. \tag{17}$$

These and similar formulae may be founded with the help of the calculation of the integral (5) by the residue theory due to the equality (2). In particular this equality on conditions that $0<|z|<\infty$, $q>p$ or $0<|z|<1$, $q=p$ and some restrictions on the parameters gives the following formula

$$G_{pq}^{mn}\left[z \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] = \sum_{k=1}^{n} \frac{\prod\limits_{j=1, j\neq k}^{m} \Gamma(b_j-b_k) \prod\limits_{j=1}^{n} \Gamma(1+b_k-a_j)}{\prod\limits_{j=1+n}^{p} \Gamma(a_j-b_k) \prod\limits_{j=1+m}^{q} \Gamma(1+b_k-b_j)} z^{b_k} \times$$

$$\times {}_pF_{q-1}\left[\begin{matrix} 1+b_k-a_1, \dots, 1+b_k-a_p; & (-1)^{p-m-n}z \\ 1+b_k-b_1, \dots, 1+b_k-b_{k-1}, 1+b_k-b_{k+1}, \dots, 1+b_k-b_q \end{matrix}\right]. \tag{18}$$

Here the symbol $_pF_q(z)$ denotes the generalized hypergeometric function

$$_pF_q\left[\begin{array}{c} a_1, \ldots, a_p; z \\ b_1, \ldots, b_q \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k \ z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k \ k!} \qquad (18)$$

If mentioned restrictions on the parameters are violated then some indefiniteness may appear in the right side of (18) and it will be changed by the combination of more complicated series containing the logarithmic function. Such logarithmic cases are connected with the existence of the multiple poles in the product (6) and the formulae of the works [1, 3] must be used for the calculation of the right side of (18) (see also the article [2], where the logarithmic case of high multipleness studied). For example, for the Macdonald function $K_\nu(z)$ with zero index those poles have only the second multipleness and the following representation takes place

$$K_0(z) = \frac{1}{2} G_{02}^{20}\left[z^2/4 \ \Big| \ 0, \ 0\right] = \frac{1}{4\pi i}\int_{\mathcal{L}} \Gamma^2(s)\left[\frac{z^2}{4}\right]^{-s} ds = \qquad (20)$$

$$= -I_0(z)\ell n(z/2) + \sum_{k=0}^{\infty} \frac{\psi(k+1)}{(k!)^2}\left[\frac{z^2}{4}\right]^k \ , \quad \psi(z) = \frac{d}{dz} \ell n \ \Gamma(z).$$

In conclusion of this part we must note that the fullest table of the particical cases of the generalized hypergeometric functions (18) is contained in the reference-book [18]. These functions for many cases of their parameters may be represented through simpler elementary and special functions.

2. The algorithm of the calculating of integrals. The following general result is proved in details and studied in the examples in the article [19] (see also the book [18, p.346]).

If some conditions of the convergence for the integral given below are fulfiled then the following equality takes place

$$\int_0^{\infty} x^{\alpha-1} \ G_{uv}^{st}\left[\sigma x \ \Big| \ \begin{array}{c}(c_u)\\(d_v)\end{array}\right] \ G_{pq}^{mn}\left[\omega x^{\ell/k} \ \Big| \ \begin{array}{c}(a_p)\\(b_q)\end{array}\right] dx = \qquad (21)$$

$$= \frac{k^{\mu} \ell^{\rho+\alpha(v-u)-1} \sigma^{-\alpha}}{(2\pi)^{b^*(\ell-1)+c^*(k-1)}} G_{kp+\ell v,kq+\ell u}^{km+\ell t,kn+\ell s} \left[ \frac{\omega^k \; k^{k(p-q)}}{\sigma^{\ell} \; \ell^{\ell(u-v)}} \middle| \begin{array}{l} \Delta(k,a_1), \; \ldots, \\ \Delta(k,b_1), \; \ldots, \end{array} \right.$$

$$\Delta(k,a_n), \; \Delta(\ell,1-\alpha-d_1), \; \ldots, \Delta(\ell,1-\alpha-d_v), \; \Delta(k,a_{n+1}), \; \ldots, \; \Delta(k,a_p)$$

$$\left. \Delta(k,b_m), \; \Delta(\ell,1-\alpha-c_1), \; \ldots, \; \Delta(\ell,1-\alpha-c_u), \; \Delta(k,b_{m+1}), \; \ldots, \; \Delta(k,b_q) \right],$$

where $c^*$, $\mu$ see in (7) and

$$b^* = s+t-(u+v)/2, \quad \rho = \sum_{j=1}^{v} d_j - \sum_{j=1}^{u} c_j + (u-v)/2 + 1 \; ,$$

$$\Delta(k,a) = \frac{a}{k}, \; \frac{a+1}{k}, \; \ldots, \; \frac{a+k-1}{k} \tag{22}$$

Thus the integral of the type (21) from the product of two G-functions is also Meijer's G-function.

Above-mentioned conditions of convergence are subdivided into 40 variants joined in special table of conditions. The variety of conditions is explained by the fact that every G-functions standing under the integral has 2 or 3 singular points (zero, infinity and, maybe, some else) in which the asymptotic behaviour of G-function has the degree, exponentially degree or oscillating character [5, 7, 13, 14] (see examples (8)-(17)). For the illustration we shall write out one variant of this special table of conditions:

$$mnst \neq 0, \quad b^*, c^* > 0, \quad \mathrm{Re}(k\alpha + kd_h + \ell b_j) > 0, \quad j=\overline{1, m}, \quad h=\overline{1, s};$$
$$\mathrm{Re}(k\alpha + kc_g + \ell a_j) < k+\ell, \quad j=\overline{1, n}, \quad g=\overline{1, t}; \quad |\arg(\omega)| < c^* \pi, \quad |\arg(\sigma)| < b^* \pi.$$

The prove of the equality (21) for the simplicity is given in the particical case $k=\ell=1$, without paying attention to the questions of carring out of all operations. Using the formula (5) we shall rewrite the integral (21) in the form of

$$\frac{1}{2\pi i} \int_{\mathscr{L}} g(\eta) \omega^{-\eta} d\eta \int_0^{\infty} x^{\alpha-\eta-1} G_{uv}^{st} \left[ \sigma x \middle| \begin{array}{l} (c_u) \\ (d_v) \end{array} \right] dx. \tag{23}$$

The inside integral is the well known Mellin transform of G-function. As the G-function (5) for straight line contour $\mathscr{L} = (\gamma - i\infty, \gamma + i\infty)$ is the inverse Mellin transform of the function (6) then the Mellin transform of G-function has the view (6). So the analogous property [11] is true for the inside integral of the formula (23)

$$\int_0^\infty x^{\alpha-\eta-1} \; G_{uv}^{s t}\left[\sigma x \;\middle|\; \begin{matrix} (c_u) \\ (d_v) \end{matrix}\right] dx = \sigma^{\eta-\alpha} \frac{\prod\limits_{j=1}^{s} \Gamma(d_j+\alpha-\eta) \prod\limits_{j=1}^{t} \Gamma(1-c_j-\alpha+\eta)}{\prod\limits_{j=s+1}^{u} \Gamma(c_j+\alpha-\eta) \prod\limits_{j=t+1}^{v} \Gamma(1-d_j-\alpha+\eta)} . \tag{24}$$

If we introduce the values (8) and (24) into (23) then we shall have the following value of the integral (21):

$$\frac{\sigma^{-\alpha}}{2\pi i} \int_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j+\eta) \prod\limits_{j=1}^{n} \Gamma(1-\alpha_j-\eta) \prod\limits_{j=1}^{s} \Gamma(d_j+\alpha-\eta) \prod\limits_{j=1}^{t} \Gamma(1-c_j-\alpha+\eta)}{\prod\limits_{j=1+n}^{p} \Gamma(a_j+\eta) \prod\limits_{j=1+m}^{q} \Gamma(1-b_j+\eta) \prod\limits_{j=1+t}^{u} \Gamma(c_j+\alpha-\eta) \prod\limits_{j=1+s}^{v} \Gamma(1-d_j-\alpha+\eta)} \left(\frac{\omega}{\sigma}\right)^{-\eta} d\eta. \tag{25}$$

According to the definition of G-function (5),(8) this value produces the right side of (21) for $k=\ell=1$.

It must be noted that the participle cases of the formulae (21) are the overwhelming majority of the integrals which one can meet in the world reference literature. The considerable part of the others integrals is the integrals which have the form (21) but include three Meijer's G-functions or which are the socalled integrals by indexes of the special functions (see, for example, [17, p.404-412]). The general formulae for their evaluation were received in the dissertation [15] and in the non-published article of O.I. Marichev. The work for their realisation on the computers is under way.

3. The calculating of the indefinite integrals. Substituting $s=u=v=k=\ell=c_1=1$, $t=d_1=0$, $\sigma=y^{-1}$ into the formula (21) and taking into account the equality (8) we shall get the value of the following indefinite integral

$$\int_0^y x^{\alpha-1} \; G_{pq}^{mn}\left[\omega x \;\middle|\; \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] dx = y^\alpha \; G_{p+1,q+1}^{m,n+1}\left[\omega y \;\middle|\; \begin{matrix} a_1,\ldots,a_n,1-\alpha,a_n,\ldots,a_p \\ b_1,\ldots,b_m,-\alpha,b_{m+1},\ldots,b_q \end{matrix}\right]. \tag{26}$$

This formula for particular values of the parameters of G-function contains the table of the simplest indefinite integrals of the elementary functions. For examples, if $m=1$, $n=p=0$, $q=2$, $\alpha=2\omega=1/2$, $b_2=0$, $b_1=1/2$, $y=z^2$ taking into account (12) it is easy to get out of (26) the trivial result

$$\int_o^y x^{-1/2}\sin\sqrt{x}\,dx = 2\int_o^* \sin t\,dt = \sqrt{\pi}\,z\,G_{4,3}^{1,1}\!\left[z^2/4\left|\begin{array}{c}1/2\\1/2,\ -1/2,\ 0\end{array}\right.\right] = \quad (27)$$

$$= z^2{}_1F_2(1;\ 2,\ 3/2;\ -z^2/4) = 2 - 2\cos z.$$

Many indefinite integrals of the elementary and special functions of the hypergeometric type are the participal cases of the formula (28). So its realization on the computers will essensially supplement above-mentioned algorithm of the calculation of indefinite integrals for elementary functions which was developed by M.Bronstein.

4. The computer's realization of the algorithm. The dialoguering regime which was realized in our INTEGRATOR system gives the possibility to make easier the first difficultest step: to elucidate the question about belonging the integrand function to participal cases of G-function (5), (6) or to product of two such cases. For this aim the special basis table of the participal cases of G-function which are wroten on the language PASCAL is used. Any man may choose necessary functions from this table and to form integrand expression of the following integral

$$\int_a^b x^{\alpha-1}G_1(\sigma x)G_2(\omega x^{\ell/k})\,dx,\quad 0\le a<b\le\infty. \quad (28)$$

The second step allows us to determine the convergence of studing integral. The convergence is understanding in the classical sense or in the sense of existence of the main part of the integral. Our programm realizes the check-up of the number conditions and put on the corresponding restrictions to the letteral parameters. This and next parts are wroten on RLISP language.

The third step realizes direct calculating of the integral by means of the substitution of the concrete values of the parameters for this case into the formula (21). Here the usual parameters counting off is realized. In the result of it the value of the integral (21) through Meijer's G-function with concrete parameters is wrotten out.

The fourth step in this case is the inverse to the first step and so it is very difficult. Received G-function is transformed here into more famous and the simplest functions if such a change is possible. Here three possible variants of actions are realized. According to the first variant the participal cases of G-function may be transformed into the elementary and special functions with the

help of special basic table the fragment of which is designated by
the formulae (8)-(17). Received special functions may permit further
simplifications. The right part of the formulae (27) and the given
bellow chain of the equalities demonstrate these operations:

$$G_{04}^{20}\left[x \ \Big| \ 1/4, \ 3/4, \ 0, \ 1/2\right] = x^{5/8} \ J_{1/2}(4x^{1/4}) =$$

$$= x^{5/8}(2/\pi)^{1/2}(4 \ x^{1/4})^{-1/2} \ \sin(4x^{1/4}) = \frac{1}{\sqrt{2\pi}} \ \sin(4x^{1/4}).$$

It should noted that the attempt of the creation of computer's
methods for the operations with the highest transcendental functi-
ons has been undertaken earlier in the connection with the working
out of the system MACSYMA. For example the algorithms of the trans-
formation of hypergeometric functions into elementary and special
functions under some conditions on the parameters have been built
[6]. These algorithms were based on the application of recurring for-
mulae. Unfortunately we don't know the results of this great work.
But we can emphasize that the approach proposed here differs from
the previous one by the highest degree of generality because it is
based on Meijer's G-function but not its participal cases, i.e. ge-
neralized hypergeometric series (18).

The second variant of actions is based on the use of the formu-
la (18) and its inverse analogue that leads to representations of
the integral (21) through the linear combinations of the generalized
hypergeometric functions. Here the conditions on the parameters ex-
cluding the logarithmic cases must be fulfiled.

The third variant gives the possibility to write out the values
of the integral (21) in the logarithmic cases when the right part of
(21) is expressed through the series and sums containing the deriva-
tives of the gamma-function and the degrees of logarithmic function.
These formulae are repsesented in the works [1, 2].

These given four steps have some else branches and additions.
For example, if one of functions from the left part of (21) takes
the fixed value of the type (8)-(14), then the programm will give
the possibility to calculate the indefinite integrals, the integrals
of fractional order and classical integral transforms of Stieltjes,
Hilbert, Laplace, Fourier, Hankel and others.

## 6. Demonstrational examples.

$$\int_o^1 x^{p-1} * (1-x**2)^{b-1} * J_v(c*x) \, dx$$

1: in "start.red"$

    INTEGRAL IS CONVERGENT UNDER CONDITIONS :

C > 0 ;
Re(B) > 0 ;
Re(P + V) > 0 ;

    THE INTERMEDIATE RESULT :

GFM( (1 1 1 3) ((-P + 2)/2) (v/2,(-v)/2,(- 2*B - P + 2)/2) C**2/4 )

    THE FINISH RESULT :

(C^V * GHF(1,2,(P + V)/2,(2*B + P + V)/2,V + 1,( - C^2 )/4)*GAMMA((P +

V)/2)*GAMMA(B))/(2*2^V *GAMMA(V + 1)*GAMMA((2*B + P + V)/2))

2: sub(p=v+2,ws);

(2^B *BESS(B + V,C)*GAMMA(B))/(2*C^B )

    Here

GFM( (1 1 1 3) (a) (b, c, d) z) = $G_{13}^{66}\left(z \ \middle| \ \begin{matrix} a \\ b, \ c \end{matrix}\right)$

GHF(1,2,a,b,c,z) = ${}_1F_2\left(\begin{matrix} a; \ z \\ b, \ c \end{matrix}\right) = \sum_{k=o}^{\infty} \frac{(a)_k \ z^k}{(b)_k (c)_k k!}$

BESS(V,Z) = $J_v(z)$

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