On Stirling numbers and Euler sums

Victor Adamchik
Wolfram Research Inc.,
100 Trade Center Dr.,
Champaign, IL 61820, USA

October 21, 1996

Abstract.
In this paper, we propose the another yet generalization of Stirling numbers of the first kind for non-integer values of their arguments. We discuss the analytic representations of Stirling numbers through harmonic numbers, the generalized hypergeometric function and the logarithmic beta integral. We present then infinite series involving Stirling numbers and demonstrate how they are related to Euler sums. Finally we derive the closed form for the multiple zeta function $\zeta(p, 1, \ldots, 1)$ for $p > 1$.

1 Introduction and notations.

Throughout this article we will use the following definitions and notations. Stirling numbers of the first kind are defined by the recurrence relation (see [1])

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right] \tag{1}$$

with initial values

$$\left[{n \atop 0}\right] = \delta_{0n}$$

which related to other notations for Striling numbers by

$$\left[{n \atop k}\right] = (-1)^{n-k} \mathrm{S}_k^{(n)} = (-1)^{n-k} \mathrm{s}(n, k)$$

The Pochhammer symbol is

$$(z)_p = \prod_{k=1}^{p} (z + k - 1) = \frac{\Gamma(z+p)}{\Gamma(z)} = \sum_{j=0}^{p} \left[{p \atop j}\right] z^j$$

The generalized hypergeometric function is defined by

$${}_p\mathrm{F}_q\left({a_1, a_2, \ldots, a_p \atop b_1, b_2, \ldots, b_q; z}\right) = \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{p} (a_m)_k z^k}{\prod_{m=1}^{q} (b_m)_k k!}$$

Following [1] we define "r-order" harmonic numbers by

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r} \tag{2}$$

or, in terms of polygamma functions,

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} \left(\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1) \right)$$

2 Harmonic numbers.

We will consider the functional equation (1) and derive its solution in terms of harmonic numbers. It is known ([2]) that

$$\begin{aligned}
\left[n \atop 1 \right] &= (n-1)! \\
\left[n \atop 2 \right] &= (n-1)!\, H_{n-1}^{(1)} \\
\left[n \atop 3 \right] &= \frac{(n-1)!}{2!} ((H_{n-1}^{(1)})^2 - H_{n-1}^{(2)})
\end{aligned}$$

To find $\left[n \atop 4 \right]$ we set $k = 4$ in (1) and using the above initial conditions we obtain

$$\left[n \atop 4 \right] = \frac{(n-1)!}{3!} ((H_{n-1}^{(1)})^3 - 3H_{n-1}^{(1)} H_{n-1}^{(2)} + 2H_{n-1}^{(3)})$$

It follows, then, that the general formula for Stirling numbers of the first kind in terms of harmonic numbers is

$$\left[n \atop m \right] = \frac{(n-1)!}{(m-1)!} w(n, m-1) \tag{3}$$

where the w-sequence is defined recursively by

$$\begin{aligned}
w(n,0) &= 1 \\
w(n,m) &= \sum_{k=0}^{m-1} (1-m)_k H_{n-1}^{(k+1)} w(n, m-1-k)
\end{aligned}$$

It is interesting to observe that for a given m the number of summation terms in the w-sequence is exactly a number of partitions of m.

The w-sequence can be rewritten also through a multiple sum

$$w(n,m) = \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n-1} \cdots \sum_{i_m=i_{m-1}+1}^{n-1} \frac{m!}{i_1\, i_2 \ldots i_m}$$

Moreover, there is another representation for the w sequence. Consider the Pochhammer function $(x-p)_p$ and find the value of its m-th derivative with respect to x at the point $p = x - 1$. Since the Pochhammer function is a ratio of Gamma functions, one can easily prove that

$$\lim_{p \to x-1} \frac{d^m}{dx^m}(x-p)_p = w(x,m)\Gamma(x) \tag{4}$$

On the other hand, taking into account that the Pochhammer symbol is the generating function for Stirling numbers, we have

$$\lim_{p \to x-1} \frac{d^m}{dx^m}(x-p)_p = \sum_{i=m}^{x} \left[{x \atop i} \right] (i-m)_m (-1)^{x-i} x^{i-1-m}$$

Equating the right sides of (4) and this identity we arrive at

$$w(n,m) = \frac{1}{(n-1)!} \sum_{i=m+1}^{n} \left[{n \atop i} \right] (i-m)_m (-1)^{n-i} n^{i-m-1} \tag{5}$$

Comparing (3) and (5) we derive the following Stirling number identity

$$\left[{n \atop m} \right] = \frac{1}{2} \sum_{i=m+1}^{n} \left[{n \atop i} \right] \binom{i-1}{m-1} n^{i-m} (-1)^{n-i}$$

where $n + m$ is an odd integer. At the end of this section we establish the link between the w-sequence and Stirling polynomials. The latter are defined (see, for example, [1]) by

$$\sigma_n(m) = \left[{m \atop m-n} \right] \frac{(m-n-1)!}{m!}$$

Comparing this definition with formula (3), we get

$$\sigma_n(m) = \frac{1}{m} w(m, m-n-1)$$

for positive integers n and m. In particular, for $n = m - 2$ this formula reduces to

$$m\,\sigma_{m-2}(m) = H^{(1)}_{m-1}$$

3 Hypergeometric functions.

Consider the hypergeometric function

$${}_{p+1}\mathrm{F}_p \left({b,\, a,\, \ldots,\, a \atop a+n, \ldots, a+n; 1} \right) \tag{6}$$

where n is a positive integer and $\Re(b) < p\,n$. We shall derive the closed form of its representation and show that this kind of hypergeometric function can be regarded as a generalization of Stirling numbers of the first kind. For this purpose we need the Mellin-Barnes contour integral that provides us with the following representation for function (6)

$$\frac{(a)_n^p}{2\pi i\,\Gamma(b)}\oint_L \frac{\Gamma(s)\,\Gamma(b-s)\,\Gamma(a-s)^p}{\Gamma(a+n-s)^p}\,(-1)^{-s}ds \tag{7}$$

where the contour of the integration L is a left loop, beginning and ending at $-\infty$ and encircling all poles of $\Gamma(s)$ in the positive direction, but none of the poles of $\Gamma(b-s)\,\Gamma(a-s)$.

Applying

$$\frac{\Gamma(b-s)}{\Gamma(b)}(-1)^{-s} = \Gamma(1-b)\Big(\frac{1}{\Gamma(1-b+s)} + \frac{e^{-\pi i b}\Gamma(b-s)}{\Gamma(s)\,\Gamma(1-s)}\Big)$$

to the integrand in (7) we obtain

$$\oint_L \frac{\Gamma(s)\,\Gamma(b-s)\,\Gamma(a-s)^p}{\Gamma(a+n-s)^p}\,(-1)^{-s}ds =$$

$$\oint_L \frac{\Gamma(s)\,\Gamma(a-s)^p}{\Gamma(1-b+s)\,\Gamma(a+n-s)^p}\,ds + e^{-\pi i b}\oint_L \frac{\Gamma(b-s)\,\Gamma(a-s)^p}{\Gamma(1-s)\,\Gamma(a+n-s)^p}\,ds$$

Observe that the second integral in the right side is zero since its integrand has no singularities inside the contour L. For the first integral we shall use the principle of the analytic continuation of Mellin-Barnes integrals which states that

$$\oint_L \frac{\Gamma(s)\,\Gamma(a-s)^p}{\Gamma(1-b+s)\,\Gamma(a+n-s)^p}\,ds = -\oint_M \frac{\Gamma(s)\,\Gamma(a-s)^p}{\Gamma(1-b+s)\,\Gamma(a+n-s)^p}\,ds$$

where M is a right loop, beginning and ending at $+\infty$ and encircling all poles of $\Gamma(a-s)$ in the negative direction, but none of the poles of $\Gamma(s)$. By using the residue theorem we obtain

Proposition 1 *If $n \in \mathbb{N}$, $b \notin \mathbb{N}$ and $\Re(b) < p\,n$ then*

$$\begin{aligned} {}_{p+1}\mathrm{F}_p\left(\begin{matrix} b,\ a,\ \ldots,\ a \\ a+n,\ldots,a+n;1\end{matrix}\right) &= (-1)^{p+1}\,(a)_n^p\,\frac{\Gamma(1-b)}{(p-1)!} \quad \times \\ \lim_{s\,\to\,a}\frac{d^{p-1}}{ds^{p-1}}&\sum_{k=0}^{n-1}\frac{\Gamma(k+s)\,(a-s)^p}{\Gamma(1-b+k+s)\,(a-s-k)_n^p}\end{aligned} \tag{8}$$

Consider two special cases of this formula. Let $n=1$, $b=a$, $a<p$ and a is not an integer. Then formula (8) becomes

$$\begin{aligned} {}_{p+1}\mathrm{F}_p\left(\begin{matrix} a,\ a,\ \ldots,\ a \\ a+1,\ldots,a+1;1\end{matrix}\right) &= \\ \frac{(-1)^{p-1}\pi\,a^p}{\sin(a\,\pi)\,(p-1)!}&w(a,p-1)\end{aligned} \tag{9}$$

If a is a positive integer then from (9) it follows that

$$
\begin{gathered}
{}_{p+1}\mathrm{F}_p\left(\begin{matrix} a,\, a,\, \ldots,\, a \\ a+1, \ldots, a+1; 1\end{matrix}\right) = \\
\frac{a^p}{(a-1)!}\sum_{k=0}^{a-1}(-1)^{a-k-1}\zeta(p-k)\left[\begin{matrix} a \\ k+1\end{matrix}\right]
\end{gathered}
\tag{10}
$$

Notice that formulas (8), (9) and (10) generalize (7.10.2.6)-(7.10.2.7) from [3]. Now we are ready to define Stirling numbers of the first kind for non-integer values of the upper argument. Combining formulas (3) and (9) we arrive at

Proposition 2 *If $\Re(z) < p$, and $p \in \mathbb{N}$ then*

$$
\left[\begin{matrix} z \\ p\end{matrix}\right] = \frac{(-1)^{p+1}}{z^p\,\Gamma(1-z)}\,{}_{p+1}\mathrm{F}_p\left(\begin{matrix} z,\, z,\, \ldots,\, z \\ z+1, \ldots, z+1; 1\end{matrix}\right)
\tag{11}
$$

or in the series form

$$
\left[\begin{matrix} z \\ p\end{matrix}\right] = \frac{(-1)^{p+1}\,\sin(\pi\, z)}{\pi}\sum_{k=0}^{\infty}\frac{\Gamma(k+z)}{k!\,(k+z)^p}
\tag{12}
$$

The following special values can be easily computed:

$$
\begin{aligned}
\left[\begin{matrix} 1/2 \\ 2\end{matrix}\right] &= -\sqrt{\pi}\,\log(4) \\
\left[\begin{matrix} 1/2 \\ 3\end{matrix}\right] &= \frac{\sqrt{\pi}}{6}\,(\pi^2 + 3\,\log^2(4)) \\
\left[\begin{matrix} 1/2 \\ 4\end{matrix}\right] &= -\frac{\sqrt{\pi}}{6}\,(\pi^2\,\log(4) + \log^3(4) + 12\,\zeta(3))
\end{aligned}
$$

Since the second argument of Stirling numbers is actually an order of the hypergeometric function, it should always be a positive integer. In the next section we find such an integral representation for Stirling numbers that allows us to generalize Stirling numbers with respect to both arguments.

4 Integral representations.

Our principal aim is to derive the analytic continuation of Striling numbers for complex values of their parameters. For this purpose we consider first the very particular case, namely $\left[\begin{matrix} n \\ 2\end{matrix}\right]$. From the section 2 follows that

$$
\left[\begin{matrix} n \\ 2\end{matrix}\right] = (n-1)!\,(\gamma + \psi(n))
\tag{13}
$$

We shall make of use the well-known integral representation of polygamma functions:

$$
\psi^{(k)}(z) - \psi^{(k)}(1) = (-1)^{k+1}\int_0^1 \frac{t^{z-1}-1}{1-t}\,\log^k(\frac{1}{t})\,dt
\tag{14}
$$

Substituting (14) into (13), we find

$$\left[{z \atop 2} \right] = \Gamma(z) \int_0^1 \frac{1 - t^{z-1}}{1 - t}\, dt, \qquad \Re(z) > 0 \tag{15}$$

In order to find the integral representation of Stirling numbers of high orders we rewrite (14) as

$$\psi^{(k)}(z) = (-1)^{k+1} \int_0^1 \frac{t^{z-1}}{1 - t} \log^k(\frac{1}{t})\, dt$$

assuming that $\Re(k) > 0$. We observe that the integrand can be obtained by differentiation with respect to the dummy variable s

$$\frac{t^{z-1}}{(1-t)^r} \log^k(\frac{1}{t}) = \lim_{s \to 0} \frac{d^k}{ds^k}((\frac{1}{t})^s \frac{t^{z-1}}{(1-t)^r})$$

We added the new parameter r here to provide the convergence of the integral at unity. Changing the order of integration and differentiation we have

$$\frac{d^k}{ds^k} \int_0^1 \frac{t^{z-s-1}}{(1-t)^s}\, dt = \frac{d^k}{ds^k}(\frac{\Gamma(1-r)\,\Gamma(z-s)}{\Gamma(1+z-s-r)})$$

If we set the variable parameter r to z, the right side will closely resemble equation (4) with $p = z - 1$ and $x = z - s$. Therefore, by equation (9) it follows that

$${}_{p+1}\mathrm{F}_p \left({z,\ z,\ \ldots,\ z \atop z+1, \ldots, z+1; 1} \right) = \frac{z^p}{(p-1)!} \int_0^1 \frac{t^{z-1}}{(1-t)^z} \log^{p-1}(\frac{1}{t})\, dt$$

However, in view of formula (11), the above integral gives us another approach to the generalization of Stirling numbers for complex values of their arguments.

Proposition 3 *If* $0 < \Re(z) < \Re(p)$, *then*

$$\left[{z \atop p} \right] = \frac{1}{\Gamma(1-z)\,\Gamma(p)} \int_0^1 \frac{t^{z-1}}{(1-t)^z} \log^{p-1}(t)\, dt \tag{16}$$

In [4] and [5] the similar integral representation has been obtained for the Stirling numbers $s(z,p)$. Though these two forms of Stirling numbers are related by

$$s(z,p) = (-1)^{z-p} \left[{z \atop p} \right]$$

where z and p are integer, however, for non-integer z and p their integral generalizations are quite different.

Now we shall verify if this integral representation (16) satisfies the functional equation (1). Beginning with $\left[{n-1 \atop k-1} \right]$ and applying an integration by parts we get

$$\left[{n-1 \atop k-1} \right] = \frac{1}{\Gamma(k-1)\,\Gamma(2-n)} \int_0^1 \frac{t^{n-2}}{(1-t)^{n-1}} \log^{k-2}(t)\, dt \; =$$

$$\frac{1}{\Gamma(k)\,\Gamma(2-n)} \int_0^1 \frac{t^{n-1}}{(1-t)^{n-1}}\, d(\log^{k-1}(t)) \; =$$

$$\frac{1}{\Gamma(k)\,\Gamma(2-n)} \left((n-1) \int_0^1 \frac{t^{n-2}}{(1-t)^{n-1}} \log^{k-1}(t)\, dt + \right.$$

$$\left. (n-1) \int_0^1 \frac{t^{n-1}}{(1-t)^{n}} \log^{k-1}(t)\, dt \right) \; =$$

$$-(n-1) \left[{n-1 \atop k} \right] + \left[{n \atop k} \right]$$

We omitted the non-integral term which is always zero assuming $\Re(k) > \Re(n) > 1$.

The integral representation (16) can be analytically continued (in the Hadamard sense) to the half-plane $\Re(z) < 0$. For simplicity we show here how to make the continuation to the strip $-1 < \Re(z) \leq 0$. For such z, the integrand in (16) has a non-integrable singularity at the point $t = 0$ which can be removed by substracting from $(1-t)^{-z}$ the first term of its Taylor's expansion. Consequently, formula (16) yields

$$\left[{z \atop p} \right] = \frac{1}{\Gamma(1-z)\,\Gamma(p)} \int_0^1 t^{z-1} \left((1-t)^{-z} - 1 \right) \log^{p-1}(t)\, dt +$$

$$\frac{1}{\Gamma(1-z)\,\Gamma(p)}\, p.f. \int_0^1 t^{z-1} \log^{p-1}(t)\, dt$$

It is easy to see that the integrand in the second integral possess the Hadamard property at the point $t = 0$:

$$p.f. \int_0^1 t^{z-1} \log^{p-1}(t)\, dt = \frac{(-1)^{p-1}}{z^p} \Gamma(p)$$

Finally we have

$$\left[{z \atop p} \right] = \frac{1}{\Gamma(1-z)\,\Gamma(p)} \int_0^1 t^{z-1} \left((1-t)^{-z} - 1 \right) \log^{p-1}(t)\, dt + \frac{(-1)^{p-1}}{z^p \Gamma(1-z)} \tag{17}$$

where $-1 < \Re(z) < \Re(p)$.

5 Series involving Stirling numbers.

In this section we consider the following type of sums involving Stirling numbers

$$\mathrm{G}_{p,q} \; = \sum_{k=p}^{\infty} \left[{k \atop p} \right] \frac{1}{k!\, k^q} \tag{18}$$

Let us begin with the simple example

$$\sum_{k=1}^{\infty} \left[{k \atop 2} \right] \frac{1}{k!\,k}$$

Using the integral representation (15) and changing the order of summation and integration, we get

$$\sum_{k=1}^{\infty} \left[{k \atop 2} \right] \frac{1}{k!\,k} = \int_0^1 \frac{\pi^2 t \, - \, 6\,\mathrm{Li}_2(t)}{6\,t\,(1-t)}\,dt = \zeta(3)$$

From this identity one would expect the pattern to remain unchanged and so that:

$$\mathrm{G}_{p,1} \, = \, \sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{1}{k!\,k} = \zeta(p+1) \tag{19}$$

To prove this formula we shall use the method of generating functions. We have to show that

$$\sum_{p=1}^{\infty} t^p \zeta(p+1) = \sum_{p=1}^{\infty} t^p \sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{1}{k!\,k}$$

The left side of this equality is a well-known sum (see [6] (54.3.1)):

$$\sum_{p=1}^{\infty} t^p \zeta(p+1) = -\gamma \, - \, \psi(1-t)$$

In the right side, changing the order of summation and taking into account the formula (52.2.1) from [6], we have

$$\sum_{p=1}^{\infty} t^p \sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{1}{k!\,k} = \sum_{k=1}^{\infty} \frac{1}{k!\,k} \sum_{p=1}^{k} t^p \left[{k \atop p} \right] = \sum_{k=1}^{\infty} \frac{(t)_k}{k!\,k} = -\gamma \, - \, \psi(1-t)$$

In a similar way, using the integral representation (15), we obtain

Proposition 4 *If* $p \in \mathbb{N}$ *, then*

$$\mathrm{G}_{2,q} = \sum_{k=2}^{\infty} \left[{k \atop 2} \right] \frac{1}{k!\,k^q} = \frac{(q+1)}{2}\zeta(q+2) - \frac{1}{q} \sum_{k=1}^{q-1} k\zeta(k+1)\zeta(q+1-k) \tag{20}$$

Infinite series of this kind can be viewed as the particular cases of the Nielsen generalized polylogarithm $\mathrm{S}_{n,p}(z)$:

$$\mathrm{S}_{n,p}(z) = \frac{1}{(n-1)!p!} \int_0^1 (-\log(t))^{n-1} (-\log(1-zt))^p \frac{dt}{t} = \sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{z^k}{k!\,k^n} \tag{21}$$

extensively studied in [7]. It was shown there, for example, that

$$\sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{z^k}{k!\,k} = \zeta(p+1) + \sum_{k=0}^{p} \frac{(-1)^{k-1}}{k!} \,\mathrm{Li}_{p+1-k}(1-z)\, \log^k(1-z) \tag{22}$$

From the polylogarithmic integral (21), performing one time integration by parts, it is easy to derive the symmetry property of Stirling sums:

$$\mathrm{G}_{p,q} = \mathrm{G}_{q,p} \tag{23}$$

Based on the integral representation (21) many interesting sums involving Stirling numbers of the first kind can be derived. Here are some of them:

$$\sum_{k=p}^{\infty} \left[{k \atop p} \right] \frac{(-1)^k}{(k+1)!} = -1 + \frac{\gamma(p+1, -\log(2))}{p!}$$

$$\sum_{k=p}^{\infty} \left[{k \atop p} \right] \frac{1}{(k+1)!} = 1$$

$$\sum_{k=3}^{\infty} \left[{k \atop 3} \right] \frac{1}{k!\, k^3} = \frac{23\,\pi^6}{15120} - \zeta(3)^2$$

$$\sum_{k=3}^{\infty} \left[{k \atop 3} \right] \frac{1}{k!\, k^4} = -\frac{\pi^4}{72}\zeta(3) - \frac{\pi^2}{3}\zeta(5) + 5\,\zeta(7)$$

$$\sum_{k=3}^{\infty} \left[{k \atop 3} \right] \frac{1}{k!\, k^5} = \frac{61\,\pi^8}{226800} + \frac{\pi^2}{12}\zeta(3)^2 - 3\,\zeta(3)\,\zeta(5)$$

In general, Stirling sums $\mathrm{G}_{p,q}$ can always be represented in finite terms of Zeta functions:

$$\mathrm{G}_{p,q} = \frac{(-1)^{q-1}}{(q-1)!\,p!} \lim_{\beta\to 0} \lim_{\alpha\to 0} \frac{d^{\,q+p-1}}{d\alpha^{\,p}\, d\beta^{\,q-1}} \frac{\Gamma(1-\alpha)\,\Gamma(1+\beta)}{\beta\,\Gamma(1-\alpha+\beta)} \tag{24}$$

The formula follows straightforwardly from the integral representation (21).

6 Euler sums.

In this section we establish a connection betwenn Stirling sums $\mathrm{G}_{p,q}$ and Euler sums. The Euler sum of the weight $e_1 + e_2 + \ldots + e_p + q$ is defined (see [9]) by

$$\mathrm{S}_{e_1, e_2, \ldots, e_p, q} = \sum_{n=1}^{\infty} \frac{H_n^{(e_1)} H_n^{(e_2)} \ldots H_n^{(e_p)}}{n^q} \tag{25}$$

We consider $\mathrm{G}_{2,n-1}$ and replace the Stirling numbers $\left[{k \atop 2} \right]$ by its representation through harmonic numbers from the section 2. We obtain

$$\mathrm{G}_{2,n-1} = \sum_{k=1}^{\infty} \left[{k \atop 2} \right] \frac{1}{(k-1)!\, k^n} = \sum_{k=1}^{\infty} \frac{H_{k-1}^{(1)}}{k^n} = \sum_{k=1}^{\infty} \frac{H_k^{(1)}}{k^n} - \sum_{k=1}^{\infty} \frac{1}{k^{n+1}} = \mathrm{S}_{1,n} - \mathrm{G}_{1,n}$$

which can be rewritten as

$$\mathrm{S}_{1,n}=1!(\mathrm{G}_{2,n-1}+\mathrm{G}_{1,n}) \tag{26}$$

In the same manner, considering $\mathrm{G}_{2,n-1}$,$\mathrm{G}_{4,n-1}$ and $\mathrm{G}_{5,n-1}$ we arrive in the very simple way (compare it with [8]) at the following identities:

$$\mathrm{S}_{1,1,n}-\mathrm{S}_{2,n}=2!(\mathrm{G}_{3,n-1}+\mathrm{G}_{2,n}) \tag{27}$$

$$\mathrm{S}_{1,1,1,n}-3\mathrm{S}_{1,2,n}+2\mathrm{S}_{3,n}=3!(\mathrm{G}_{4,n-1}+\mathrm{G}_{3,n}) \tag{28}$$

$$\mathrm{S}_{1,1,1,1,n}-6\mathrm{S}_{1,1,2,n}+3\mathrm{S}_{2,2,n}+8\mathrm{S}_{1,3,n}-6\mathrm{S}_{4,n}=4!(\mathrm{G}_{5,n-1}+\mathrm{G}_{4,n}) \tag{29}$$

The pattern is quite obvious. The coefficients by S in the left sides are identical to those which are by harmonic numbers in representations of Stirling numbers from the section 1.

Now let us consider the multiple Euler zeta sum (also called Euler/Zagier sums)

$$\zeta(s_1,s_2,\ldots,s_d)=\sum_{n_1>n_2>\ldots>n_d}\frac{1}{n_1^{s_1}n_2^{s_2}\ldots n_d^{s_d}}$$

intensively studied in recent times ([10], [11], [12], [13], [14]) and establish the closed form representation for $\zeta(p,1,1,\ldots,1)$, where $p>1$.

It was known to Euler that

$$\zeta(p,1,\ldots,1)=\mathrm{G}_{n+1,p-1} \tag{30}$$

However, it is been only few years when the closed form for $\zeta(p,1,1)$ was discovered (see [11], [8]). Here let us demonstrate another (and very simple) approach to evaluating of $\zeta(p,1,1)$. We have

$$\zeta(p,1,1)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\sum_{n_3=1}^{n_2-1}\frac{1}{n_1^p n_2 n_3}$$

Evaluating the inner sum, we obtain

$$\zeta(p,1,1)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{H_{n_2}^{(1)}}{n_1^p n_2}-\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{1}{n_1^p n_2^2}$$

or

$$\zeta(p,1,1)=\zeta(2+p)-\mathrm{S}_{2,p}+\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{H_{n_2}^{(1)}}{n_1^p n_2} \tag{31}$$

since

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{1}{n_1^p n_2^2}=\sum_{n_1=1}^{\infty}\frac{H_{n_1-1}^{(2)}}{n_1^p}=\zeta(2+p)-\mathrm{S}_{2,p}$$

where S is defined by (25). To our good fortune, the inner sum in the right side of the identity (31) is summable in terms of harmonic numbers.

We need the following lemma which can be proved by using the integral representation (14)

Lemma 1
$$\sum_{k=1}^{p}\frac{H_k^{(1)}}{k}=\frac{1}{2}((H_p^{(1)})^2+H_p^{(2)}) \tag{32}$$

Thus, the identity (35) can be rewritten

$$\zeta(p,1,1)=\zeta(2+p)-\mathrm{S}_{2,p}-\mathrm{S}_{1,p+1}+\frac{1}{2}\mathrm{S}_{1,1,p}+\frac{1}{2}\mathrm{S}_{2,p}$$

and therefore, taking into account formulas (20), (26) and (27), we have

$$\zeta(p,1,1)=\mathrm{G}_{3,p-1} \tag{33}$$

where G is defined by (24). As we know, for non-integer values of p the Stirling sum G is defined by the Nielsen generalized polylogarithm (21). Since that, the above identity can be rewritten as

$$\zeta(p,1,1)=\mathrm{S}_{3,p-1}(1) \tag{34}$$

which generalizes Markett's formula for non-integers p.

Now let us approach $\zeta(p,1,1,1)$. Proceeding in the similar way, we obtain

$$\zeta(p,1,1,1)=\mathrm{S}_{3,p}-\zeta(p+3)-\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{H_{n_2}^{(1)}}{n_1^p n_2^2}+\frac{1}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{(H_{n_2}^{(1)})^2}{n_1^p n_2}+\frac{1}{2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1-1}\frac{H_{n_2}^{(2)}}{n_1^p n_2^2} \tag{35}$$

It is unknown to me whether or not the above inner sums are doable in finite terms. Fortunately, they are doable in pairs. We need the following lemmas:

Lemma 2
$$\sum_{k=1}^{p}\frac{H_k^{(2)}}{k}+\sum_{k=1}^{p}\frac{H_k^{(1)}}{k^2}=H_p^{(3)}+H_p^{(1)}H_p^{(2)} \tag{36}$$

This lemma is almost obvious. To prove it you need to consider any one of the above sums, replace harmonic numbers by the finite sum (2) and change the order of summation.

Lemma 3

$$\sum_{k=1}^{p}\frac{(H_k^{(1)})^2}{k}+\sum_{k=1}^{p}\frac{H_k^{(2)}}{k}=\frac{1}{3}((H_p^{(1)})^3+3H_p^{(1)}H_p^{(2)}+2H_p^{(3)}) \tag{37}$$

The lemma can be easily proved by induction.

Now, substituting (36) and (37) into (35) and taking into account formulas (26),(27) and (28), we immediately arrive at

Proposition 5

$$\zeta(p,1,1,1) = \mathrm{G}_{4,p-1} = \mathrm{S}_{4,p-1}(1) \tag{38}$$

Comparing formulas (30), (33) and (38), we can derive the general representation for $\zeta(p,1,\ldots,1)$

Proposition 6

$$\zeta(p,1,\ldots,1) = \mathrm{G}_{n+1,p-1} = \mathrm{S}_{n+1,p-1}(1) \tag{39}$$

where n is a number of 1's in arguments of the multiple zeta function.

Acknowledgement.

I would like to thank R. Crandall and B. Gosper for helpful discussions.

References

[1] R. L. Graham, D. E. Knuth, O. Patashnik (1989), *Concrete Mathematics*, Addison-Wesley.

[2] L. Comtet (1974),*Advanced Combinatorics*,Reichel, Dordrecht and Boston.

[3] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev (1990), *Integrals and Series*, Vol. 3: More Special Functions, Gordon and Breach, New York.

[4] P.L.Butzer, M.Hauss (1991), *Stirling numbers of first and second kind*, In "Approximation, Interpolation and Summability", Israel Math Conf. Proceedings, Weizmann Press, Israel, Vol. 4, 1991, 89-108.

[5] P.L.Butzer, M.Hauss (1992), *Riemann zeat function*, Appl. Math. Lett. 5(2), 1992, 83-88.

[6] E. R. Hansen (1975),*A Table of Series and Products*, Prentice-Hall, Inc., N.J.

[7] K.S.Kolbig (1986),*Nielsen's generalized polylogarithms* , SIAM J. Math. Anal., 17 (5), p. 1232-1256.

[8] D. Borwein, J. Borwein, R. Girgensohn (1995),*Explicit evaluation of Euler sums*, Proceedings of the Edinburgh Mathematical Society 38, 277-294.

[9] F. Flajolet, B. Salvy (1996), *Euler Sums and Contour Integral Representations*, Experimental Mathematics (to appear).

[10] D. Zagier (1994), *Values of zeta functions and their applications*, preprint, Max-Planck.

[11] C. Markett (1994), *Triple Sums and the Riemann Zeta Function*, J. Number Theory, 48, p. 113-132.

[12] R.E. Crandall, J. P. Buhler (1994), *On the Evaluation of Euler Sums*, Experimental Mathematics **3**, 275-285.

[13] R.E. Crandall (1996), *Fast evaluation of multiple zeta sums*, Experimental Mathematics (to appear).

[14] J. M. Borwein, D. M. Bradley, D. J. Broadhurst (1996), *Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k*, manuscript.