A certain series associated with Catalan's constant

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Abstract

A parametric class of series generated by integration of complete elliptic integrals

$$\sum_{\substack{k=0 \\ k\neq -r}}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)\,16^k}$$

is evaluated in closed form. Alternative proofs to results of Ramanujan and others are given. A particular case of the Saalschutzian hypergeometric series ${}_4F_3(1)$ is derived.

Keywords: summation, elliptic functions, hypergeometric functions, Catalan's constant

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0. Preamble

The subject of our interest is the hypergeometric series generated by elliptic integrals

$$S(r)=\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(k+r)\,16^k}=\frac{1}{r}\,{}_3F_2\Big(\frac{1}{2},\,\frac{1}{2},\,r;\,1,\,r+1;\,1\Big) \tag{1}$$

This series has a long and interesting story. About a century ago Ramanujan [1, p. 351 and 2, p. 39] in his first letter to Hardy stated without proof a particular case of (1), when the parameter $r=n$ is a positive integer, namely

$$S(n)=\frac{16^n}{\pi\,n^2\binom{2n}{n}^2}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{16^k} \tag{2}$$

In 1927, when Ramanujan's collected papers were published and result (2) became publicly known, it attracted a great deal of attention. Different proofs were given by Watson [3] and Darling [4], later Bailey [5] and Hodgkinson [6] generalized (2) to

$${}_3F_2(a,\,b,\,c+n-1;\,c,\,a+b+n;\,1)=\frac{\Gamma(n)\,\Gamma(a+b+n)}{\Gamma(a+n)\,\Gamma(b+n)}\sum_{k=0}^{n-1}\frac{(a)_k\,(b)_k}{k\,!\,(c)_k} \tag{3}$$

which gives Ramanujan's result when $a=b=\frac{1}{2}$ and $c=1$. Ramanujan [7, pp. 237-239 and 2, p. 45] also stated a complementary formula to (2), when the parameter $r=n+\frac{1}{2}$ is a half integer, namely

$$S\Big(n+\frac{1}{2}\Big)=\frac{4}{\pi}\,\frac{\binom{2n}{n}^2}{16^n}\left(2\,G+\sum_{k=0}^{n-1}\frac{16^k}{\binom{2k}{k}^2(2\,k+1)^2}\right). \tag{4}$$

Here G is Catalan's constant defined by

$$G=\frac{1}{2}\int_0^1\mathbf{K}(k)\,d\,k=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2\,k+1)^2}$$

and $\mathbf{K}$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) = \int_0^{\frac{\pi}{2}} \frac{d\,t}{\sqrt{1 - k^2 \sin^2 t}}$$

As mentioned in [2, p. 47], Ramanujan's proofs of formulas (2) and (4) most likely were based on the recurrence equation

$$\left(r + \frac{1}{2}\right)^2 S(r+1) - r^2 S(r) = \frac{1}{\pi} \tag{5}$$

subject to initial conditions. This equation is derived from the fact that $S(r)$ is generated by integration of complete elliptic integrals as

$$S(r) = \frac{2}{\pi} \int_0^1 z^{r-1}\, \mathbf{K}(z)\, d\,z, \qquad \operatorname{Re}(r) > 0. \tag{6}$$

In 1981, unawared of Ramanujan's equation (5), Dutka [8] employed by (6) rediscoverd formulas (2) and (4). In section 3 we outline the derivation of equation (5), as well as it's solution. In view of (5), it's pretty straightforward to see that for any rational $r = n + p$, where n is a positve integer and $0 < p \leq 1$, series (1) has a closed form representation

$$S(n+p) = \frac{{(p)_n}^2}{{(p+\frac{1}{2})_n}^2} \left(S(p) + \frac{1}{\pi\, p^2} \sum_{k=0}^{n-1} \frac{{(p+\frac{1}{2})_k}^2}{{(p+1)_k}^2} \right) \tag{7}$$

Here $(p)_n = p(p+1)\ldots(p+n-1)$ is the Pochhammer symbol. There are only three known cases when the function $S(p)$ is expressible in terms other than hypergeometric functions, namely $p = 1, \frac{1}{2}, \frac{1}{4}$.

$$S(1) = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 1, 2; 1\right) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1\right) = \frac{4}{\pi}$$

$$S\left(\frac{1}{2}\right) = 2\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1\right) = \frac{8\,G}{\pi}$$

$$S\left(\frac{1}{4}\right) = 4\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}; 1, \frac{5}{4}; 1\right) = \frac{\Gamma(\frac{1}{4})^4}{4\,\pi^2}$$

where $\Gamma(z)$ is the Euler gamma function. All these cases are due to Ramanujan. L.Glasser [9] made a conjection that it is possible to express $S(\frac{1}{2^k})$ for $k \geq 3$ in finite terms, however that is remained to be seen.

It does not appear to have been previously studied the case when the parameter r in (1) is a negative integer (assuming that the term $r = -k$ is dropped from summation):

$$S(r)=\sum_{\substack{k=0\\k\neq -r}}^{\infty}\frac{\binom{2k}{k}^2}{(k+r)\,16^k}. \tag{8}$$

A few particular cases of (8) appeared in the handbooks by E.P. Adams and R.L. Hippisley [10] and by E.R. Hansen [11]:

$$S(-1)=-\frac{2G+1}{\pi}+\log(2)-\frac{1}{2}$$
$$S(-2)=-\frac{18G+13}{16\pi}+\frac{9}{16}\log(2)-\frac{21}{64}$$

In the present paper, using contour integration technique, we will show that for negative integer r, sum (8) is solvable in closed form by

$$S(r)=-S\Big(\frac{1}{2}-r\Big)+\frac{4}{16^{-r}}\binom{-2r}{-r}^2(H_{-r}-H_{-2r}+\log 2),\quad r=0,\,-1,\,-2,\,\ldots \tag{9}$$

where H_n are the harmonic numbers $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

As a consequence of this result, in section 2, we derive the new representation for Saalschutzian ${}_4F_3(1)$ series with special set of the parameters

$$\Big(n-\frac{1}{2}\Big)\,{}_4F_3\Big(1,\,1,\,n+\frac{1}{2},\,n+\frac{1}{2};\,2,\,n+1,\,n+1;\,1\Big)=$$
$$\frac{4n^2}{2n-1}(H_{n-1}+\log 4)-\frac{16^n}{\binom{2n}{n}^2}\,{}_3F_2\Big(\frac{1}{2},\,\frac{1}{2},\,n-\frac{1}{2};\,1,\,n+\frac{1}{2};\,1\Big) \tag{10}$$

1. Evaluation

We consider two cases, when r is positive and negative We denote $S^{+}(r)=S(r)$ for $\mathrm{Re}(r)>0$ and $S^{-}(r)=S(r)$ for $\mathrm{Re}(r)\le 0$.

Let r be a positive integer. We transform series (1) to a definite integral involving complete elliptic integrals. Multiplying the summand by x^{k+r} and differentiating it with respect to x, we get

$$g(r,\,x)=x^{r-1}\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{x^k}{16^k}=\frac{2}{\pi}\,x^{r-1}\,\mathbf{K}(x),\quad |x|<1. \tag{11}$$

where $\mathbf{K}(x)$ is the elliptic integral. Integrating both sides of (11), we arrive at

$$S^{+}(r) = \int_0^1 g(r, x)\, dx = \frac{2}{\pi} \int_0^1 x^{r-1}\, \mathbf{K}(x)\, dx, \quad Re(r) > 0. \tag{12}$$

In the next subsections we evaluate $S^{+}(r)$, by first developing a recurrent equation for $S^{+}(r)$, and then solving it by iteration. The result depends on the disparity of r.

Now let us consider the second case when r is a negative integer. We split the series $S(r)$ into two sums as

$$S^{-}(r) = \sum_{\substack{k=0 \\ k \neq -r}}^{\infty} \frac{\binom{2k}{k}^2}{(k+r)\, 16^k} = \left(\sum_{k=0}^{-r-1} + \sum_{k=-r+1}^{\infty} \right) \frac{\binom{2k}{k}^2}{(k+r)\, 16^k}.$$

Leaving the first sum unchanged, and converting the second sum into an elliptic integral (by applying the same reasoning as above), we obtain

$$S^{-}(r) = \sum_{k=0}^{-r-1} \frac{\binom{2k}{k}^2}{(k+r)\, 16^k} + \int_0^1 x^{r-1} \left(\frac{2}{\pi}\, \mathbf{K}(x) - \sum_{k=0}^{-r} \binom{2k}{k}^2 \frac{x^k}{16^k} \right) dx, \tag{13}$$

$$Re(r) \leq 0$$

In subsection 1.3, using the contour integration technique, we establish a functional relation between $S^{-}(r)$ into $S^{+}(r)$.

1.1 $S^{+}(r)$ for r a non-negative integer

Consider the system of indefinite integrals

$$\begin{cases} k_p(x) = \int x^p\, \mathbf{K}(x)\, dx \\ e_p(x) = \int x^p\, \mathbf{E}(x)\, dx \end{cases} \tag{14}$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, the above integral system can be reduced to the system of coupled recurrent equations

$$\begin{cases} k_p(x) = x^p\, k_0(x) - 2\, p\, (k_p(x) - k_{p-1}(x) + e_{p-1}(x)) \\ e_p(x) = x^p\, e_0(x) - \frac{2}{3}\, p\, (e_{p-1}(x) + e_p(x) + k_p(x) - k_{p-1}(x)) \end{cases}$$

with initial conditions

$$\begin{cases} 2\,k_0(x) = \mathbf{E}(x) + (x-1)\,\mathbf{K}(x) \\ \frac{3}{2}\,e_0(x) = (x+1)\,\mathbf{E}(x) + (x-1)\,\mathbf{K}(x) \end{cases}$$

Eliminating $e_{p-1}(x)$ from the first equation, and $k_{p-1}(x)$ and $k_p(x)$ from the second, the system is simplified to

$$\begin{cases} k_p(x) = \frac{4\,p^2}{(2\,p+1)^2}\,k_{p-1}(x) + \frac{2\,x^p\,\mathbf{E}(x) + 2\,(2\,p+1)\,(x-1)\,x^p\,\mathbf{K}(x)}{(2\,p+1)^2} \\ e_p(x) = \frac{4\,p^2}{(2\,p+1)\,(2\,p+3)}\,e_{p-1}(x) + \frac{2\,(1-2\,p+(2\,p+1)\,x)\,x^p\,\mathbf{E}(x) + 2\,(x-1)\,x^p\,\mathbf{K}(x)}{(2\,p+1)\,(2\,p+3)} \end{cases}$$

Now we compute the values of $k_p(x)$ and $e_p(x)$ at the limiting points $x = 0$ and $x = 1$. We get two recurrent equations

$$\begin{aligned} k_p(1) &= \frac{4\,p^2}{(2\,p+1)^2}\,k_{p-1}(1) + \frac{2}{(2\,p+1)^2}\,,\ \ p \geq 1 \\ k_p(0) &= 0\,,\ \ p \geq 0 \\ k_0(1) &= 2. \end{aligned} \tag{15}$$

and

$$\begin{aligned} e_p(1) &= \frac{4\,p^2}{(2\,p+1)\,(2\,p+3)}\,e_{p-1}(1) + \frac{4}{(2\,p+1)\,(2\,p+3)}\,,\ \ p \geq 1 \\ e_p(0) &= 0\,,\ \ p \geq 0 \end{aligned} \tag{16}$$

In view of formulas (12) and (15), we conclude that

$$S^+(r) = \frac{2}{\pi}\,(k_{r-1}(1) - k_{r-1}(0)) = \frac{2}{\pi}\,k_{r-1}(1) \tag{17}$$

where $S^+(r)$ satisfies the recurrence relation

$$\begin{aligned} \left(r + \frac{1}{2}\right)^2 S^+(r+1) - r^2\,S^+(r) &= \frac{1}{\pi},\ \ r \geq 1 \\ S^+(1) &= \frac{4}{\pi}. \end{aligned} \tag{18}$$

Equation (18) can be solved by iteration (see section 3 for details). We have proven

Proposition 1.1 *Let n be a positive even. Then $S(n)$ defined by (1) evaluates to*

$$S(n) = \frac{16^n}{\pi\, n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \binom{2k}{k}^2 \frac{1}{16^k}. \tag{19}$$

1.2 $S^+(r)$ for r a positive half-integer

Consider slightly different (than (14)) system of indefinite integrals

$$\begin{cases} \hat{k}_p(x) = \int x^{p-\frac{1}{2}}\, \mathbf{K}(x)\, d\, x \\ \hat{e}_p(x) = \int x^{p-\frac{1}{2}}\, \mathbf{E}(x)\, d\, x \end{cases} \tag{20}$$

where the parameter p is a positive integer or zero, and $\mathbf{E}(x)$ and $\mathbf{K}(x)$ are complete elliptic integrals. Using integration by parts, we transform (20) to the system of recurrent equations

$$\begin{cases} p^2\, \hat{k}_r(x) = (p - \tfrac{1}{2})^2\, \hat{k}_{p-1}(x) + \tfrac{1}{2}\, x^{p-\frac{1}{2}} \Big(\mathbf{E}(x) + 2\, p\, (x-1)\, \mathbf{K}(x)\Big) \\ p\,(p+1)\, \hat{e}_r(x) = (p - \tfrac{1}{2})^2\, \hat{e}_{p-1}(x) + x^{p-\frac{1}{2}} \Big((p\,(x-1)+1)\, \mathbf{E}(x) + \tfrac{x-1}{2}\, \mathbf{K}(x)\Big) \end{cases} \tag{21}$$

where

$$\hat{k}_0(x) = \pi \sqrt{x}\; {}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\Big)$$

$$\hat{e}_0\,(x) = \pi \sqrt{x}\; {}_3F_2\Big(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\Big)$$

and ${}_3F_2(x)$ is the hypergeometric function. By computing the limits at $x = 0$ and $x = 1$, system (21) yields (assuming $p > 0$)

$$\begin{aligned} \hat{k}_p(1) &= \frac{(p-\frac{1}{2})^2}{p^2}\, \hat{k}_{p-1}(1) + \frac{1}{2\, p^2}, \quad p \geq 1 \\ \hat{k}_p(0) &= 0\,, \quad p \geq 0 \\ \hat{k}_0(1) &= 4\, G \end{aligned} \tag{22}$$

where G is Catalan's constant. Therefore,

$$S^+\Big(p + \frac{1}{2}\Big) = \frac{2}{\pi}\, \hat{k}_p(1)\,, \quad p \geq 0 \tag{23}$$

The sequence $S^+(r)$, where r is a positive half integer, satisfies the same recurrence equation (18) , but with a different initial condition:

$$\left(r+\frac{1}{2}\right)^2 S^+(r+1) - r^2 S^+(r) = \frac{1}{\pi}$$
$$S^+\left(\frac{1}{2}\right) = \frac{8\,G}{\pi} \tag{24}$$

Solving this recurrence by iteration (see section 3 for details), we have proven

Proposition 1.2 *Let* n *be a positive integer. Then* $S(n+\frac{1}{2})$ *defined by (1) evaluates to*

$$S\left(n+\frac{1}{2}\right) = \frac{4}{\pi}\,\frac{\binom{2\,n}{n}^2}{16^n}\left(2\,G + \sum_{k=0}^{n-1}\frac{16^k}{\binom{2\,k}{k}^2 (2\,k+1)^2}\right) \tag{25}$$

1.3 $S^-(r)$, for r a negative integer

Recall formula (13). Observing that the finite sum inside of the integrand

$$\sum_{k=0}^{-r}\binom{2\,k}{k}^2\frac{x^k}{16^k}$$

is the Taylor expansion of $\frac{2}{\pi}\,\mathbf{K}(x)$ at $x=0$, we pull that sum out of integration, by understanding integration in the Hadamard sense (*finite part*). Computing limits at the end points and obliterating logarithmic and polynomial order singularities, we get

$$S^-(r) = f.\,p.\frac{2}{\pi}\int_0^1 x^{-r-1}\,\mathbf{K}(x)\,d\,x.$$

Comparing this integral with formula (12) immediately implies that

$$S^-(r) = S^+(r) + F(r)$$

where $F(r)$ is an unknown function. The necessity of F becomes obvious once we recall that in the original series we skip the term $k=-r$, when r is a negative integer. In order to find F, we derive a contour integral representation for the sum $S(r)$ as

$$S(r) = \frac{1}{2\,\pi\,i}\int_{\gamma-i\,\infty}^{\gamma+i\,\infty}\frac{\Gamma(s)\;\Gamma(\frac{1}{2}-s)}{\Gamma(1-s)\,\Gamma(\frac{1}{2}+s)}\;\frac{d\,s}{(r-s)}. \tag{26}$$

The contour $(\gamma - i\infty,\ \gamma + i\infty)$ is a straight line lying in the strip $0 < \gamma = Re(s) < \frac{1}{2}$. Integral (26) evaluates to (1) by summing residues at single poles s = 0, -1, -2, ... , lying to the left of the contour. However, if r is a negative integer, the integrand in (26) has a double pole at $s = r$. According to the definition of $S(r)$ we must skip this pole. Thus, we have

$$S^{-}(r) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\ \Gamma(\frac{1}{2} - s)}{\Gamma(1-s)\,\Gamma(\frac{1}{2}+s)}\ \frac{ds}{(r-s)} - \underset{s=r}{res}\left(\frac{\Gamma(s)\ \Gamma(\frac{1}{2} - s)}{\Gamma(1-s)\,\Gamma(\frac{1}{2}+s)}\ \frac{1}{(r-s)}\right) \tag{27}$$

As a matter of fact, the contour integral herein can also be computed via residues at the poles $s = \frac{1}{2},\ \frac{3}{2},\ \ldots$, lying to the right of the contour. Evaluating the integral via those poles allows us to avoid the double pole at $s = r$. This yields

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\ \Gamma(\frac{1}{2} - s)}{\Gamma(1-s)\,\Gamma(\frac{1}{2}+s)}\ \frac{ds}{(r-s)} = -\sum_{k=0}^{\infty} \frac{(2k)!^2}{k!^4\,(k - r + \frac{1}{2})\,16^k} = -S^{+}\left(\frac{1}{2} - r\right). \tag{28}$$

Finally, computing the residue

$$\underset{s=r}{res}\left(\frac{\Gamma(s)\ \Gamma(\frac{1}{2} - s)}{\Gamma(1-s)\,\Gamma(\frac{1}{2}+s)}\ \frac{1}{(r-s)}\right) = \frac{4}{16^{-r}} \binom{-2r}{-r}^2 \left(H_{-2r} - H_{-r} - \log 2\right)$$

we establish

Proposition 1.3 *Let r be a negative integer or zero. Then*

$$S^{-}(r) = -S^{+}\left(\frac{1}{2} - r\right) - \frac{4}{16^{-r}} \binom{-2r}{-r}^2 \left(H_{-r} - H_{-2r} + \log 2\right) \tag{29}$$

where $S^{+}(\frac{1}{2} - r)$ is defined in Proposition 1.2.

1.4 $S^{-}(r)$ for r a negative half integer

This case immediately follows from the previous subsection, taking into consideration that the integrand in (26) has only a single pole at $s = r$.

Proposition 1.4 *Let n be a positive integer. Then*

$$S^{-}\left(-n+\frac{1}{2}\right) = -S^{+}(n). \tag{30}$$

2. Special cases of hypergeometric functions

In this section we derive a particular case of the Saalschutzian hypergeometric series ${}_4F_3\,(1)$. We begin by recalling that the hypergeometric series ${}_{p+1}F_p(a_1 \ldots,\ a_{p+1};\, b_1 \ldots,\, b_p;\, 1)$ is called Saalschutzian if parameters a_i and b_i satisfy the relation

$$1 + a_1 + a_2 + \ldots + a_{p+1} = b_1 + \ldots + \ b_p$$

Proposition 2.1 *Let n be a positive integer. Then*

$$\frac{(2\,n-1)^2}{8\,n^2}\ {}_4F_3\left(1,\, 1,\, n+\frac{1}{2},\, n+\frac{1}{2};\, 2,\, n+1,\, n+1;\, 1\right) = -\frac{4\,G}{\pi} + H_{n-1} + \log 4 - \frac{2}{\pi}\sum_{k=0}^{n-2}\frac{16^k}{(2\,k+1)^2\binom{2\,k}{k}^2} \tag{31}$$

where G is Catalan's constant, and H_n are harmonic numbers.

Proof.

In view of formula (29) with $r=-n,\ n\ =\ 0,\ 1,\ 2,\ \ \ldots$, we have

$$S^{-}(-n) = -S^{+}\left(n+\frac{1}{2}\right) - \frac{4}{16^n}\binom{2\,n}{n}^2\left(H_n - H_{2\,n} + \ \log 2\right) \tag{32}$$

where $S^{+}(n+\frac{1}{2})$ is defined in (25). On the other hand, if we evaluate the original sum (8) by means of the hypergeometric function, we obtain

$$S^{-}(-n) = \sum_{k=0}^{n-1}\frac{\binom{2\,k}{k}^2}{(k-n)\,16^k} + \frac{\binom{2\,n+2}{n+1}^2}{16^{n+1}}\ {}_4F_3\left(1,\, 1,\, n+\frac{3}{2},\, n+\frac{3}{2};\, 2,\, n+2,\, n+2;\, 1\right). \tag{33}$$

The finite sum in the right-hand side of (33) can be evaluated in terms of harmonic numbers (see Proposition 3.2) as

$$16^n \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k (n-k)} = 4 \binom{2n}{n}^2 \sum_{k=0}^{n-1} \frac{1}{2k+1} = 2 \binom{2n}{n}^2 \left(2 H_{2n-1} - H_{n-1}\right) \quad (34)$$

Combining formulas (32) and (33), and replacing n by $n-1$, we arrive at (31).

Remark. By using different ideas, formula (31) was first proved in [13].

3. Addendum

In this section we provide a solution to equations (18) and (24)

Proposition 3.1 *The solutions to the recurrence relation*

$$\begin{gathered} (2n+1)^2 x_{n+1} - (2n)^2 x_n = a, \\ x_1 = b \end{gathered} \quad (35)$$

is

$$x_n = \frac{16^n}{4 n^2 \binom{2n}{n}^2} \left(b + a \sum_{k=1}^{n-1} \frac{\binom{2k}{k}^2}{16^k} \right) \quad (36)$$

Proof.

We solve recurrence (35) by iteration. Iterating it $n-1$ times, we get

$$x_{n+1} = b \prod_{j=0}^{n-1} \frac{(2n-2j)^2}{(2n+1-2j)^2} + a \sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n-2j)^2}{\prod_{j=0}^{k} (2n+1-2j)^2}. \quad (37)$$

In pretty straightforward manner the finite products herein can be converted to the binomial coefficients by using Euler's product representation for the Gamma function. We obtain

$$\prod_{j=0}^{n-1} \frac{(2n-2j)}{(2n-2j+1)} = \frac{4^{n+1}}{2(n+1)\binom{2n+2}{n+1}}$$

and

$$\sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2\,n-2\,j)^2}{\prod_{j=0}^{k} (2\,n-2\,j+1)^2} = \frac{16^{n+1}}{4\,(n+1)^2 \binom{2\,n+2}{n+1}^2} \sum_{k=1}^{n} \frac{\binom{2\,k}{k}^2}{16^k}.$$

Substituting them into (37) yields the desired result.

Proposition 3.2 *Let n be a positive integer. Then*

$$\frac{16^n}{4\binom{2\,n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2\,k}{k}^2}{16^k\,(n-k)} = \sum_{k=0}^{n-1} \frac{1}{2\,k+1}. \tag{38}$$

Proof.

We rearrange the terms in the sum in the left-hand side of (38), by summing them in the opposite order from
$n-1$ to 0. We get

$$\sum_{k=0}^{n-1} \frac{\binom{2\,k}{k}^2}{(n-k)\,16^k} = \sum_{k=1}^{n} \frac{\binom{2\,n-2\,k}{n-k}^2}{k\,16^{n-k}}.$$

Since the summand evaluates to zero for $k > n$, we extend the range of summation to infinity. Using the definition of the hypergeometric series, we rewrite that sum in terms of ${}_4F_3$ as

$$\frac{16^n}{4\binom{2\,n}{n}^2} \sum_{k=1}^{\infty} \frac{\binom{2\,n-2\,k}{n-k}^2}{k\,16^{n-k}} = \frac{n^2}{(2\,n-1)^2}\,{}_4F_3\!\left(1,\,1,\,1-n,\,1-n;\,2,\,\frac{3}{2}-n,\,\frac{3}{2}-n;\,1\right)$$

The latter further simplifies to polygamma functions by formula 7.5.3.43 from [12] as

$$\frac{2\,n^2}{(2\,n-1)^2}\,{}_4F_3\!\left(1,\,1,\,1-n,\,1-n;\,2,\,\frac{3}{2}-n,\,\frac{3}{2}-n;\,1\right) = \psi\!\left(n+\frac{1}{2}\right) - \psi\!\left(\frac{1}{2}\right) = \sum_{k=0}^{n-1} \frac{2}{2\,k+1}$$

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