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Finite and Infinite Series

Victor Adamchik

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1 Abstract

This tutorial is intended as an introduction to symbolic summation within *Mathematica*. It will talk about possibilities of the package `Algebra`SymbolicSum``. The tutorial will not assume detailed knowledge of the *Mathematica* system, but knowledge of the mathematics is needed.

The topics covered will include:

1. Introduction to the algorithms
2. Relations to `Integrate` and `DSolve`
3. Future directions

It will be nice if you bring some ideas about what new kinds of sums you would like to see in the next versions of *Mathematica*.

2 Purpose

The aim of this tutorial is to demonstrate the use of *Mathematica* to evaluate finite and infinite sums. These are supported by the package `Algebra`SymbolicSum`` that was mentioned for the first time in *Mathematica* Version 2.0.

The first section includes a brief introduction to the main algorithm implemented within the package. It will provide many examples. This section also contains a discussion about divergence (in the usual sense) of series.

The next section illustrates some relations between `Sum` on one side and `Integrate` and `DSolve` on the other side. It will show that in some cases the definite integration and `DSolve` can return infinite sums as the answer.

The last section discusses new kinds of sums that will be implemented in the future versions of *Mathematica*.

3 Introduction to the Algorithm

The general method implemented for the evaluation of the series is to represent it in terms of hypergeometric functions and then to use the classical summation theorems such as Gauss, Saalschutz, Kummer, Dixon and other theorems from the theory of hypergeometric functions. It is a well-known fact that many sum identities can be justified within the theory of hypergeometric functions. Here is the canonical form of the generalized hypergeometric function:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{z^k \prod_{j=1}^{p} (a_j)_k}{k! \prod_{j=1}^{q} (b_j)_k} \tag{1} $$

where $(a)_k$ is the Pochhammer symbol

$$ (a)_k = a\ (a+1) \ldots (a+k-1) \tag{2} $$

On other hand, the relation (1) is not only the notation but it is an analytical continuation. In essence, the function from the left side of (1) continues analytically the series from right side. Hypergeometric functions (as well as other special functions) may have many different forms of the representation. For example, one of them is the Mellin-Barnes contour integral:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{1}{2\,\pi\, i}\,\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\oint_L \frac{\Gamma(s)\prod_{j=1}^{p}\Gamma(a_j - s)\,(-z)^{-s}\,ds}{\prod_{j=1}^{q}\Gamma(b_j - s)}, \tag{3} $$

which is an analytical continuation of the series (1) to the domain $p = q + 1$, $|z| \geq 1$. This representation is widely used in the theory of hypergeometric functions.

■ 3.1 Sums with rational polynomials

We will presume if the series can be expressed in hypergeometric functions, it is evaluated. In this section we will not discuss the question of how to simplify hypergeometric functions. Let us start with a simple example:

$$ \sum_{k=0}^{\infty} \frac{1}{5 + 14\,k + 8\,k^2} \tag{4} $$

In the first step we need to factorize the denominator of the summand

$$ 5 + 14\,k + 8\,k^2 = (1 + 2\,k)\;(5 + 4\,k) $$

Then represent every monom by means of the Pochhammer symbol

$$ \frac{1}{b + a\,k} = \frac{(\frac{b}{a})_k}{b\,(1 + \frac{b}{a})_k} \tag{5} $$

In the next step we add $k!$ to the denominator and its Pochhammer representation $(1)_k$ to the numerator. Now the sum (4) has the following form:

$$ \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k\,(1)_k\,(\frac{5}{4})_k}{5\,k!\,(\frac{3}{2})_k\,(\frac{9}{4})_k} \tag{6} $$

that is the generalized hypergeometric function

$$ \frac{{}_3F_2(\frac{1}{2}, 1, \frac{5}{4}; \frac{3}{2}, \frac{9}{4}, 1)}{5} \tag{7} $$

This is the *Mathematica* output:

```
In[1]:= <<Algebra`SymbolicSum`

In[2]:= Sum[ 1/(5 + 14k + 8 k^2), {k, 0, Infinity}]

        2   Pi   Log[2]
Out[2]= - - -- - ------
        3   12     6
```

■ 3.2 Factorial sums

Now we will consider the sum with factorial functions

$$ \sum_{k=0}^{\infty} \frac{x^k\;(2\,k)!}{k!^2} \tag{8} $$

and transform it to the hypergeometric function using steps similar to those in the previous section. For that we rewrite $(2\,k)!$ in terms of Pochhammer symbols. By using the relation between the factorial and Gamma function

$$k! = \Gamma(k+1), \tag{9}$$

Legendre's duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}\,\Gamma(z)\,\Gamma(\frac{1}{2}+z)}{\sqrt{\pi}} \tag{10}$$

and the recurrence relation for Gamma function

$$\Gamma(n+z) = \Gamma(z)\,(z)_n, n = 0, 1, \ldots, \tag{11}$$

we find

$$(2k)! = 4^k\,(\frac{1}{2})_k\,k! \tag{12}$$

Thus we have obtained from (8) the following series

$$\sum_{k=0}^{\infty} \frac{4^k\,x^k\,(\frac{1}{2})_k}{k!} \tag{13}$$

which is the hypergeometric function

$${}_1F_0(\frac{1}{2}; 4x) \tag{14}$$

according to the formula (1).

This is the *Mathematica* output:

```
In[3]:= Sum[x^k (2k)!/(k!)^2, {k, 0, Infinity}]

                 1
Out[3]= ----------------
        Sqrt[1 - 4 x]
```

■ 3.3 Binomial sums

Here we will show that binomial sums can be expressed in hypergeometric functions too. Take the series from the February issue of *Mathematics Magazine* [1]

$$\frac{3}{16} + 9 \sum_{k=1}^{\infty} \frac{(4k^2-1)\,\binom{2k-2}{k-1}^2}{4\,2^{4k}\,k^2\,(k+1)^2} \tag{15}$$

By using formulas (9) through (12) and

$$\Gamma(z-k) = \frac{(-1)^k\,\Gamma(z)}{(1-z)_k)} \tag{16}$$

we find

$$\binom{2k-2}{k-1} = -\frac{2^{2k-1}\,k\,(-\frac{1}{2})_k}{(1)_k} \tag{17}$$

Therefore, the sum (15) can be rewritten

$$\frac{3}{16} + 9 \sum_{k=1}^{\infty} \frac{(4k^2-1)\,(-\frac{1}{2})_k^{\,2}}{16\,(k+1)^2\,(1)_k^{\,2}} \tag{18}$$

and finally it reduces to the hypergeometric function

$$\frac{3}{16} + \frac{27\,{}_3F_2(1, \frac{1}{2}, \frac{5}{2}; 3, 3, 1)}{256} \tag{19}$$

Now let us simplify the above hypergeometric function. We will do that by two steps. First, lower the order of that hypergeometric function. We will use this result

$$\begin{aligned} {}_pF_q(a_1, \ldots, a_{p-1}, 1; b_1, \ldots, b_{q-1}, n; z) = \frac{(n-1)\,\prod_{j=1}^{q-1}(b_j-1)}{z\,\prod_{j=1}^{p-1}(a_j-1)} \\ \left({}_pF_q(a_1-1, \ldots, a_{p-1}-1, 1; b_1-1, \ldots, b_{q-1}-1, n-1; z) - 1\right), \end{aligned} \tag{20}$$

for the integer and the positive n. Appling this relation to (19) two times, we obtain

$$\frac{3\,{}_2F_1(-\frac{3}{2},\frac{1}{2};1;1)}{4} \tag{21}$$

On the next step we shall make use of the Gauss theorem [2]:

$${}_2F_1(a,b;c,1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{22}$$

It follows that the expression (21) converts to the form:

$$\frac{3\,\Gamma(1)\,\Gamma(2)}{4\,\Gamma(\frac{1}{2})\,\Gamma(\frac{5}{2})} \tag{23}$$

Thus we find that the sum (15) is $\frac{1}{\pi}$.

This is the *Mathematica* output:

```
In[4]:= Expand[3/16 + 9 Sum[ (4k^2-1) Binomial[2k-2, k-1]^2/
                    (4 2^(4k) k^2 (k+1)^2), {k, Infinity} ]]

         1
Out[4]= --
        Pi
```

■ 3.4 Finite series

In this section we shall demonstrate that the theory of hypergeometric functions can also be used to evaluate finite series. Consider the following sum that was published in *Mathematics Magazine* as the problem N1389 [3]:

$$\sum_{k=0}^{n} (-3)^k \binom{2\,n}{2\,k} \tag{24}$$

At first we need to express the binomial function in terms of Pochhammer symbols. According to formulas (10) through (12) and

$$\Gamma(x-k) = \frac{(-1)^k\,\Gamma(x)}{(1-x)_k} \tag{25}$$

we find

$$\binom{2\,n}{2\,k} = \frac{(\frac{1}{2}-n)_k\,(-n)_k}{k!\,(\frac{1}{2})_k} \tag{26}$$

Thus the sum (24) transforms to the following form

$$\sum_{k=0}^{n} \frac{(-3)^k\,(\frac{1}{2}-n)_k\,(-n)_k}{k!\,(\frac{1}{2})_k} \tag{27}$$

Note that, for $k > n$ the Pochhammer symbol $(-n)_k$ is zero. It follows that the top limit in the series can be changed from n to ∞ and the sum can be written as

$$\sum_{k=0}^{\infty} \frac{(-3)^k\,(\frac{1}{2}-n)_k\,(-n)_k}{k!\,(\frac{1}{2})_k} \tag{28}$$

Hence, in the notation of the hypergeometric function (1) it follows immediately

$${}_2F_1(\frac{1}{2}-n,-n;\frac{1}{2};-3) \tag{29}$$

which is a Gegenbauer polynomial, and the sum of (24) is

$$4^n\,\cos(2\,n\,\arctan(\sqrt{3})) \tag{30}$$

Consider another example of the finite series:

$$\sum_{k=0}^{n-r} \binom{n}{r+k} \binom{n+r+k}{k} (-2)^{(-k)} \tag{31}$$

In a similar way, we can express this series in terms of the Gauss function:

$$\frac{n!}{(n-r)!\,r!}\,{}_2F_1(r-n, n+r+1, r+1, \frac{1}{2}) \tag{32}$$

Apply the following identity that is usually known as Euler's identity [2] to the expression (32):

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z) \tag{33}$$

We obtain

$$\frac{n!}{(n-r)!\,r!}\,{}_2F_1(1+n, -n, r+1, \frac{1}{2})\left(\frac{1}{2}\right)^{(-r)} \tag{34}$$

Finally, we shall use the Bailey's theorem [2]:

$${}_2F_1(a, 1-a, c, \frac{1}{2}) = \frac{\Gamma(\frac{1}{2}+\frac{c}{2})\,\Gamma(\frac{c}{2})}{\Gamma(\frac{1}{2}-\frac{a}{2}+\frac{c}{2})\,\Gamma(\frac{a}{2}+\frac{c}{2})} \tag{35}$$

Therefore, we find the sum of (31) is

$$\frac{\sqrt{\pi}\,n!}{(n-r)!\,\Gamma(\frac{1}{2}-\frac{n}{2}+\frac{r}{2})\,\Gamma(1+\frac{n}{2}+\frac{r}{2})} \tag{36}$$

This is the *Mathematica* output:

```
In[5]:= Sum[(-2)^(-k) Binomial[n,r+k]
        Binomial[n+r+k,k],{k,0,n-r}]

                   Sqrt[Pi] n!
Out[5]= -----------------------------------
                 n + r            1 - n + r
        (n - r)! (-----)! Gamma[---------]
                   2                 2
```

One more example that was published in *The American Mathematical Monthly* as the problem N10223 [4]. Prove that

$$\sum_{k=n}^{2n-1}\left((1-p)^n p^{k-n} + (1-p)^{k-n} p^n\right)\binom{k-1}{n-1} = 1 \tag{37}$$

We shall show that it can be proved within the theory of hypergeometric functions. Take the answer that *Mathematica* returns:

```
In[6]:= Sum[ ((1-p)^n p^(k-n) + (1-p)^(k-n) p^n) Binomial[k-1,
        n-1],{k, n, 2 n-1}]

Out[6]= (2 Gamma[n] Gamma[1 + n] -

         n  n
  (1 - p)  p  Gamma[2 n] Hypergeometric2F1[1, 2 n, 1 + n, 1 - p] -

         n  n
  (1 - p)  p  Gamma[2 n] Hypergeometric2F1[1, 2 n, 1 + n, p])/

  ((-1 + n)! Gamma[1 + n])
```

The Gauss function ${}_2F_1(a,b,c,z)$ is the analytic function in the whole complex plain of z. And the analytic continuation of ${}_2F_1(a,b,c,z)$ from $|z|<1$ to the domain $|1-z|<1$ is given by the following formula:

$${}_2F_1(a,b,c,p) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,{}_2F_1(a,b,a+b-c+1,1-p) + \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-p)^{c-a-b}\,{}_2F_1(c-a,c-b,c-a-b+1,1- \tag{38}$$

Applying this rule and the following rules

```
In[7]:= %6/.{w_! :> Gamma[w+1],Gamma[-n] :>
        Gamma[1-n]/(-n),Gamma[n+1] :> n Gamma[n]}//Expand
```

to `Out[6]`, we get

```
Out[7]= 1
```

It is interesting to build another sums like the above (37). I found the following two sum identities that can be proved by the similar way:

$$\sum_{k=n}^{\infty}\left((1-p)^n p^{k-n}+(1-p)^{k-n}p^n\right)\binom{k-1}{n-1}=2 \tag{39}$$

$$\sum_{k=2n}^{\infty}\left((1-p)^n p^{k-n}+(1-p)^{k-n}p^n\right)\binom{k-1}{n-1}=1 \tag{40}$$

The first identity follows straightforwardly from *Mathematica*:

```
In[8]:= Sum[ Binomial[k-1, n-1] (p^n q^(k-n) + p^(k-n)
        q^n),{k,n,Infinity}]/.{w_! :> Gamma[w+1]}

Out[8]= 2
```

The second identity follows from (37) and (39).

3.5 Divergent series

Here we shall discuss some problems with the formal conversion of a series to hypergeometric functions. The problem is the representation of a series in terms of hypergeometric functions is the analytical continuation of that series. Thus, if we consider a series, just as the series, it is possible that for some values of the parameters a series will be divergent. On the other hand, that series as the function can be continued analytically to the domain of values of those parameters. Consider the series

$$\sum_{k=0}^{\infty} x^k \tag{41}$$

It is easy to see the series is divergent for $|x| \geq 1$. *Mathematica* returns the following

The output is the analytic function for any $x \neq 1$ and moreover this function is the analytical continuation of the series (41) to the domain $|x| \geq 1$.

```
In[9]:= Sum[x^k,{k,0,Infinity}]

            1
Out[9]= -----
        1 - x
```

Take another example that is odd:

$$\sum_{k=1}^{\infty} \sin(k\,x) \tag{42}$$

This series is convergent for only $x = 0$. *Mathematica* returns:

```
In[10]:= Sum[Sin[k x],{k,1,Infinity}]

          I        I x
          - (1 + E    )
          2
Out[10]= --------------
                  I x
            -1 + E
```

It is not hard to understand how *Mathematica* returned that output. For that we need to recall the Euler formula:

$$\sin(k\,x) = \frac{-i}{2}\left(e^{i\,k\,x} - e^{-i\,k\,x}\right)$$

Therefore, the series (42) can be rewritten as

$$\frac{-i}{2}\left(\sum_{k=1}^{\infty} e^{i\,k\,x} - \sum_{k=1}^{\infty} e^{-i\,k\,x}\right)$$

Every sum here is the geometric series (41). And the above *Mathematica* output is correct if $|e^{i\,x}| < 1$ and $|e^{i\,x}| > 1$. So far *Mathematica* is not checking conditions for convergence and that's why we see the `Out[10]`. There is no other way to eliminate such cases as to make the code of convergence tests for the series.

4 Relations to the Integrate and DSolve

In this section we will discuss the relations between `Sum`, `Integrate` and `DSolve`. In *Mathematica* Version 2.1, `Integrate` and `DSolve` can return infinite sums as the answer. In some cases the package ``Algebra`SymbolicSum` `` can handle those sums, and in other cases it can not. Here we will consider a couple of examples.

4.1 Differential equations

In Version 2.1 there are a few new packages. One package is ``Calculus`DSolve` `` that is destined to extend the possibilities of the internal `DSolve`. This package is capable of handling some classes of linear and non-linear differential equations. Consider the linear differential equation with the variable coefficients:

$$3\,y(x) + (5\,x+1)\;y'(x) + x\;(x+1)\;y''(x) = 0 \tag{1}$$

Load the `DSolve` package

```
In[1]:= <<Calculus`DSolve`

In[1]:= <<Calculus`DSolve`
```

`DSolve` returns the infinite sum with the dummy variable *Integrate'var* that we changed to *var*.

```
In[2]:= DSolve[ x(x+1) y''[x] + (5x+1) y'[x] +3 y[x]==0 , y[x],
        x] /.Integrate`var -> var

Out[2]= {{y[x] ->
    2 C[1]                -2 Log[-x]
  ---------- + C[2] (------------ +
          3                    3
   (1 + x)              (1 + x)

                    var           1            1       var
          2 Sum[((-1)    (-(---------)   - ---------) x
                            1 + var      2 + var

               Pochhammer[3, var]) / var!, {var, 0, Infinity}])}}
```

This series can be evaluated by the `SymbolicSum` package.

Load the package

```
In[3]:= <<Algebra`SymbolicSum`
```

and it follows straightforwardly

```
In[4]:= %2

                    2 C[1]                   3 + x        2 Log[-x]
Out[4]= {{y[x] -> ---------- + C[2] (-(----------) - ------------)}}
                          3                    2              3
                   (1 + x)             (1 + x)        (1 + x)
```

4.2 Definite integration

In *Mathematica* Version 2.1 the definite integration has been improved, too. Consider one of the new integrals

$$\int_0^\infty \frac{\log(1+\frac{1}{\sqrt{x}})\,\log(1+x)}{x^{\frac{3}{2}}}\,dx \tag{2}$$

`Integrate` returns the following answer:

```
In[1]:= Integrate[(Log[1+x^(-1/2)] Log[1+x])/x^(3/2),{x,0,
        Infinity}]/.Integrate`var->var

                         2
                       Pi      Pi Log[4]
Out[1]= -2 Pi + --- + --------- bD — +
                        2          2

                    var    1          1                1          1
   Sum[((-1)    (——————— + —— bD ——————— + ——————— + ——————— bD -)
                     1               1                   1             1 + var
                     - - var   -(-) + var   - + var
                     2               2                   2

                       1                                   2
        Pochhammer[-, var] Pochhammer[1, var] ) /
                       2

                                3
     (var! Pochhammer[-, var] Pochhammer[2, var]),
                                2

    {var, 0, Infinity}]
```

where the dummy variable *Integrage'var* was changed to *var*. By using the `SymbolicSum` package, we find

```
In[2]:= <<Algebra`SymbolicSum`

In[3]:= %1

                                              2
                                      5 Pi     Pi Log[4]
Out[3]= 4 Catalan - 2 Pi + ------ + ---------
                                       12          2
```

Using the summation package we can extend the integrate function. Consider the example:

$$\int_0^\infty \frac{x^{2n+1}}{e^{\pi x}-1}dx \tag{3}$$

Mathematica cannot handle this integral. But it is not hard to do that. Note the integral (3) can be rewritten as:

$$\int_0^\infty \frac{x^{2n+1}\left(\sum_{k=0}^{\infty} e^{-(k\pi x)}\right)}{e^{\pi x}}dx \tag{4}$$

Change the order of integration and summation:

$$\sum_{k=0}^{\infty}\int_0^\infty \frac{x^{2n+1}}{e^{(k+1)\pi x}}dx \tag{5}$$

Now by using *Mathematica* we can evaluate that expression:

```
In[4]:= Sum[ Integrate[x^(1+2 n) E^(-Pi x
        (k+1)),{x,0,Infinity}]//PowerExpand,{k,0,Infinity}]

            2 - 2 n
Out[4]= Pi          Gamma[2 + 2 n] Zeta[2 + 2 n]
```

5 Future Directions

In future versions of *Mathematica*, there are plans to extend the present package in at least two directions. One of them is to allow *Mathematica* to handle series that are elliptic functions. For example,

$$\sum_{n=0}^{\infty} q^{n^2}\cos(2\,n\,u) \tag{1}$$

which is the Theta function. The second involves series with special functions such as `PolyGamma` and `Zeta`:

$$\sum_{k=1}^{\infty}\frac{\Psi(k)}{k^2} \tag{2}$$

Also there are plans to enable the code to distinguish divergent and convergent series. At the high end, it would see possibilities to find conditions of the absolute convergence, the conditional convergence, or uniform convergence.

References

[1] J.A.Ewell. "The Catalan Numbers and π" *Mathematics Magazine* 1992(**65**), No.1, pp. 36-37.

[2] L.J.Slater. *Generalized hypergeometric functions*. Cambridge, 1965.

[3] Problems. N1389, *Mathematics Magazine* 1992(**65**), No.1, p. 57.

[4] Problems. N10223, *The American Mathematical Monthly* 1992(99), No.5, p. 462.