SOME SERIES OF THE ZETA AND RELATED FUNCTIONS

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Abstract. We propose and develop yet another approach to the problem of summation of series involving the Riemann Zeta function $\zeta(s)$, the (Hurwitz's) generalized Zeta function $\zeta(s,a)$, the Polygamma function $\psi^{(p)}(z)$ $(p = 0, 1, 2, \cdots)$, and the polylogarithmic function $\mathrm{Li}_s(z)$. The key ingredients in our approach include certain known integral representations for $\zeta(s)$ and $\zeta(s,a)$. The method developed in this paper is illustrated by numerous examples of closed-form evaluations of series of the aforementioned types; the method developed in Section 2, in particular, has been implemented in *Mathematica* (Version 3.0). Many of the resulting summation formulas are believed to be new.

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1. Introductions, Definitions, and Preliminaries

A rather classical (over two centuries old) theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), was revived in 1986 by Shallit and Zikan [23] as the following problem:

$$\sum_{\omega\in\mathcal{S}} (\omega-1)^{-1} = 1, \tag{1.1}$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathcal{S} := \left\{n^k : n, k \in \ \setminus \{1\}\ \ (:= \{1, 2, 3, \cdots\})\right\} \quad . \tag{1.2}$$

In terms of the Riemann Zeta function $\zeta(s)$ defined by (see, for details, Titchmarsh [26] and Ivić [17])

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1), \end{cases} \tag{1.3}$$

which can indeed be continued analytically to the whole complex s-plane except for a simple pole at $s = 1$ with residue 1, Goldbach's theorem (1.1) assumes the elegant form (*cf.* [23, p. 403]):

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1 \tag{1.4}$$

or, equivalently,

$$\sum_{k=2}^{\infty} \mathcal{F}(\zeta(k)) = 1, \tag{1.5}$$

where $\mathcal{F}(x) := x - [x]$ denotes the *fractional* part of $x \in$. As a matter of fact, it is fairly straightforward to observe also that

$$\sum_{k=2}^{\infty} (-1)^k \mathcal{F}(\zeta(k)) = \frac{1}{2}, \tag{1.6}$$

$$\sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k)) = \frac{3}{4}, \quad \text{and} \quad \sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k+1)) = \frac{1}{4}.l \tag{1.7}$$

Another remarkable result involving the Riemann's ζ-function is the following series representation for $\zeta(3)$:

$$\zeta(3) = \frac{\pi^2}{7}\left\{1 - 4\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\,2^{2k}}\right\} \tag{1.8}$$

or, equivalently,

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\,2^{2k}}, \tag{1.9}$$

since $\zeta(0) = -\frac{1}{2}$. The series representation (1.8) is contained in a 1772 paper by Leonhard Euler (1707–1783) (see, *e.g.*, Ayoub [2, pp. 1084–1085]). It was rediscovered by Ramaswami [21] and (more recently) by Ewell [11]. (See also Srivastava [25, p. 7, Equation (2.23)] where Euler's result (1.8) was reproduced *actually* from the work of Ramaswami [21].) Numerous further series representations for $\zeta(3)$, which are analogous to (1.8) or (1.9), can be found in the works of Wilton [28], Zhang and Williams ([29] and [30]), Cvijović and Klinowski [7], and others (*cf.*, *e.g.*, Tsumura [27, p. 384] and Ewell [13, p. 1004]; see also Berndt [4]).

A considerably large variety of methods were used in the aforementioned works, and also in the works of (among others) Jensen [18], Dinghas [8], Srivastava ([24] and [25]), Klusch [20], and Choi *et al.* ([5] and [6]), dealing with summation of series involving the Riemann ζ-function and its such extensions as the (Hurwitz's) generalized Zeta function $\zeta(s,a)$ defined usually by (*cf.* [26, p. 36])

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{1.10}$$

$$(\Re(s) > 1; \quad a \neq 0, -1, -2, \cdots),$$

which, just as $\zeta(s) = \zeta(s,1)$, is meromorphic everywhere in the complex s-plane with a simple pole at $s = 1$ with residue 1. The main object of this paper is to present yet another approach to the problem of summation of series involving Zeta and related functions.

In our present investigation, we shall also make use of the Polygamma function $\psi^{(p)}(z)$ defined by

$$\psi^{(p)}(z) = \frac{d^{p+1}}{dz^{p+1}} \{\log \Gamma(z)\} \qquad (p \in {}_0 := \cup \{0\}) \quad , \tag{1.11}$$

the polylogarithmic function $\mathrm{Li}_s(z)$ defined by

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad (\Re(s) > 1), \tag{1.12}$$

the generalized harmonic numbers $H_n^{(s)}$ defined by (*cf.* Graham *et al.* [15])

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s} \qquad (n \in ; \;\; s \in), \tag{1.13}$$

the Bernoulli numbers B_n defined by (*cf.*, *e.g.*, Erdélyi *et al.* [9, p. 35])

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi), \tag{1.14}$$

and the Stirling numbers of the second kind $S(n,k)$ defined by (*cf.* Riordan [22])

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k!\, S(n,k), \tag{1.15}$$

which indeed satisfy the recurrence relation [22, p. 226]:

$$S(n,k) = S(n-1,k-1) + k\, S(n-1,k) \qquad [S(n,1) := 1]. \tag{1.16}$$

Some of these functions are related to one another as noted below:

$$\begin{aligned} &\mathrm{Li}_1(z) = -\log(1-z); \quad \mathrm{Li}_s(1) = \zeta(s) = \zeta(s,1); \\ &\mathrm{Li}_s(-1) = (2^{1-s}-1)\,\zeta(s); \end{aligned} \tag{1.17}$$

$$\psi^{(p)}(z) = (-1)^{p+1}\, p!\, \zeta(p+1,z) \qquad (p \in {}_0; \quad z \neq 0,-1,-2,\cdots) \quad ; \tag{1.18}$$

$$H_n^{(s)} = \zeta(s) - \zeta(s,n+1) \qquad (\Re(s) > 1; \quad n \in \;) \quad . \tag{1.19}$$

Furthermore, we have

$$\zeta(0,a) = \tfrac{1}{2} - a; \quad \zeta(s,2) = \zeta(s) - 1; \quad \zeta\left(s,\tfrac{1}{2}\right) = (2^s - 1)\,\zeta(s); \tag{1.20}$$

$$\zeta(s,a+n) = \zeta(s,a) - \sum_{k=0}^{n-1} \frac{1}{(k+a)^s} \qquad (n \in \;)$$

and, for $\psi(z) = \psi^{(0)}(z)$,

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{z+k} \qquad (n \in \;; \quad \psi(1) = -\gamma) \quad , \tag{1.21}$$

where γ denotes the Euler-Mascheroni constant (see, for example, Abramowitz and Stegun [1]).

2. Evaluation of the Zeta Sums

We begin by illustrating the method of evaluation of the sums of series involving Zeta functions. The key ingredients in our approach happen to include the familiar integral representations (*cf.*, *e.g.*, [26]):

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt \qquad (\Re(s) > 1) \tag{2.1}$$

and

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, e^{-(a-1)t}}{e^t - 1}\, dt \qquad (\Re(s) > 1;\ \Re(a) > 0)\,. \tag{2.2}$$

Consider the sum

$$\Omega(a) = \sum_{k=1}^\infty f(k)\, \zeta(k+1, a) \qquad (\Re(a) > 0)\,, \tag{2.3}$$

where the sequence $\{f(n)\}_{n=1}^\infty$ is assumed to possess a generating function:

$$F(t) = \sum_{k=1}^\infty f(k)\, \frac{t^k}{k!} \tag{2.4}$$

and

$$f(n) = O\left(\frac{1}{n}\right) \qquad (n \to \infty). \tag{2.5}$$

Upon replacing the Zeta function in (2.3) by its integral representation given by (2.2) with $s = k+1$, if we invert the order of summation and integration, we obtain

$$\Omega(a) = \int_0^\infty F(t)\, \frac{e^{-(a-1)t}}{e^t - 1}\, dt, \tag{2.6}$$

where we have also made use of the generating function (2.4).

Thus the problem of summation of series of the type (2.3) has been reduced formally to that of integration in (2.6). Although the integral in (2.6) appears to be fairly involved for symbolic integration, yet it may be possible to reduce it to (2.1) or (2.2) (or another known integral), especially when $F(t)$ is a power, exponential, trigonometric or hyperbolic function.

Our first example (Proposition 1 below) would illustrate how this technique actually works.

PROPOSITION 1. *Let n be a positive integer. Then*

$$\sum_{k=1}^\infty \frac{(-1)^k}{k} \{\zeta(nk) - 1\} = \log\left(\prod_{j=0}^{n-1} \Gamma\left(2 - (-1)^{(2j+1)/n}\right)\right). \tag{2.7}$$

REMARK 1. In Proposition 1, as also in Equations (2.9), (2.10), and (2.12) below, it is tacitly assumed that

$$-1 = e^{\log(-1)},$$

where $\log z$ denotes the principal branch of the logarithmic function in the complex z-plane for which

$$-\pi < \arg(z) \leq \pi \qquad (z \neq 0).$$

Proof. Denote, for convenience, the left-hand side of the summation formula (2.7) by $\Theta(n)$. Then, in view of the second relationship in (1.20), we can apply the integral representation (2.2) with $s = nk$ and $a = 2$. Upon inverting the order of summation and integration, which can be justified by the absolute convergence of the series and the integral involved, we thus find that

$$\Theta(n) = n \int_0^\infty \frac{dt}{te^t(e^t-1)} \sum_{k=1}^\infty \frac{(-t^n)^k}{\Gamma(nk+1)}. \tag{2.8}$$

By recognizing the series in (2.8) as a trigonometric function of order n (see, for details, Erdélyi *et al.* [10, Section 18.2]) or, alternatively, by using an easily derivable special case of a known result [16, p. 207, Entry (10.49.1)], we have

$$\sum_{k=1}^\infty \frac{(-t^n)^k}{\Gamma(nk+1)} = -1 + \frac{1}{n} \sum_{j=1}^n \exp\left(t(-1)^{(2j+1)/n}\right). \tag{2.9}$$

Substituting from (2.9) into (2.8), and inverting the order of summation and integration once again, we obtain

$$\begin{aligned}\Theta(n) = \lim_{r\to 1-} \Bigg\{ &-n \int_0^\infty \frac{dt}{t^r\, e^t(e^t-1)} \\ &+ \sum_{j=1}^n \int_0^\infty \frac{\exp\left(-\left[1-(-1)^{(2j+1)/n}\right] t\right)}{t^r(e^t-1)}\, dt \Bigg\},\end{aligned} \tag{2.10}$$

where the parameter $r < 1$ is inserted with a view to providing convergence of the integrals at their lower terminal $t = 0$.

Finally, we evaluate each integral in (2.10) by means of (2.2) and proceed to the limit as $r \to 1-$. Indeed, by making use of the known behavior of $\zeta(1-r, a)$ near $r = 1$ for fixed a (*cf.*, *e.g.*, Erdélyi *et al.* [9, p. 26]):

$$\zeta(1-r,a) \sim \frac{1}{2} - a + (1-r)\log\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) \qquad (r \to 1-;\ a \text{ fixed}), \tag{2.11}$$

we arrive at the right-hand side of the assertion (2.7).

In precisely the same manner, we can prove a *mild* generalization of Proposition 1, which we state here as

PROPOSITION 2. *Let n be a positive integer. Then*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\,\zeta(nk,a) &= -n\log\Gamma(a)\\ &\quad+\log\left(\prod_{j=0}^{n-1}\Gamma\left(a-(-1)^{(2j+1)/n}\right)\right) \qquad (\Re(a)\geq 1)\,.\end{aligned}\tag{2.12}$$

Next we prove

PROPOSITION 3. *Let n be a positive integer. Then, in terms of the Bernoulli numbers B_n defined by* (1.14) *and the Stirling numbers $S(n,k)$ defined by* (1.15),

$$\begin{aligned}\sum_{k=2}^{\infty}(-1)^k\ \{\zeta(k)-1\}\ k^n &= -1+\frac{1-2^{n+1}}{n+1}\,B_{n+1}\\ &\quad-\sum_{k=1}^{n}(-1)^k\,k!\ \zeta(k+1)S(n+1,k+1).\end{aligned}\tag{2.13}$$

Proof. Making use of the integral representation (2.2) with $a=2$, if we invert the order of summation and integration, and then evaluate the resulting integral and sum, we find that

$$\begin{aligned}\Lambda(n) &:= \sum_{k=2}^{\infty}(-1)^k\ \{\zeta(k)-1\}\ k^n\\ &= \lim_{r\to 1}\left(r\,\frac{d}{dr}\right)^n\{r\left[\psi(r+2)-\psi(2)\right]\}\,.\end{aligned}\tag{2.14}$$

Since

$$\left(r\,\frac{d}{dr}\right)^r\{r\ f(r)\} = \sum_{k=0}^{n} S(n+1,k+1)\,f^{(k)}(r)\ r^{k+1}\tag{2.15}$$

and

$$\psi^{(k)}(3) = (-1)^k\ k!\ \left\{1+2^{-k-1}-\zeta(k+1)\right\} \qquad (k\in\ ),\tag{2.16}$$

we find from (2.14) that

$$\begin{aligned}\Lambda(n) &= -1-\sum_{k=1}^{n}(-1)^k\ k!\ \zeta(k+1)\,S(n+1,k+1)\\ &\quad+\sum_{k=0}^{n}(-1)^k\ k!\ 2^{-k-1}\ S(n+1,k+1).\end{aligned}\tag{2.17}$$

In order to complete the proof of Proposition 3, we thus need to show that

$$\sum_{k=0}^{n} (-1)^k \; k! \; 2^{-k-1} \, S(n+1, k+1) = \frac{1 - 2^{n+1}}{n+1} \, B_{n+1} \tag{2.18}$$

in terms of the Bernoulli numbers B_n defined by (1.14). In fact, it is known that [16, p. 351, Entry (52.2.36)]

$$\sum_{k=1}^{n} (-1)^k \; k! \; 2^{-k} \, S(n, k) = \frac{2}{n+1} \, (1 - 2^{n+1}) \, B_{n+1}, \tag{2.19}$$

which, in view of the recurrence relation (1.16) with n and k replaced by $n+1$ and $k+1$, respectively, yields the desired identity (2.18).

Similarly, we can prove

PROPOSITION 4. *Let n be a positive integer. Then*

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} \; k^n = 1 + \sum_{k=1}^{n} k! \; \zeta(k+1) \, S(n+1, k+1). \tag{2.20}$$

The following list provides further summation formulas involving series of Zeta functions, which can be derived by applying the foregoing technique.

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{k+2} = \frac{2}{3} - \frac{\gamma}{2} + \log 2 + 6\zeta'(-1); \tag{2.21}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+1)(k+2)} = \frac{1-\gamma}{6} - 2\zeta'(-1); \tag{2.22}$$

$$\sum_{k=2}^{\infty} \frac{k^2}{k+1} \, \{\zeta(k) - 1\} = \frac{3}{2} - \frac{\gamma}{2} + \frac{\pi^2}{6} - \frac{1}{2} \log(2\pi); \tag{2.23}$$

$$\sum_{k=2}^{\infty} \frac{k^2}{k+1} \, \{\zeta(2k+1) - 1\} = \frac{9}{16} - \gamma + \log 2 - \frac{1}{2} \, \zeta(3); \tag{2.24}$$

$$\sum_{k=1}^{\infty} \{\zeta(4k) - 1\} = \frac{7}{8} - \frac{\pi}{4} \, \coth \pi; \tag{2.25}$$

$$\sum_{k=1}^{\infty} (-1)^k \, \{\zeta(4k) - 1\} = 1 + \frac{\pi}{2\sqrt{2}} \, \frac{\sin(\pi\sqrt{2}) + \sinh(\pi\sqrt{2})}{\cos(\pi\sqrt{2}) - \cosh(\pi\sqrt{2})}; \tag{2.26}$$

$$(2.27) \qquad \sum_{k=1}^{\infty} \{\zeta(4k) - 1\}\, z^{4k} = \frac{3z^4 - 1}{2(z^4 - 1)} - \frac{\pi z}{4}\{\cot(\pi z) + \coth(\pi z)\} \qquad (|z| < 2)\,,$$

which contains both (2.25) and (2.26) as limiting cases;

$$(2.28) \qquad \begin{aligned} \sum_{k=1}^{\infty} \{\zeta(2k) - 1\}\, \sin k = -\frac{1}{2}\cot\left(\frac{1}{2}\right) \\ + \frac{\pi}{2}\, \frac{\sin\left(\frac{1}{2}\right)\sin\left(2\pi\cos\left(\frac{1}{2}\right)\right) - \cos\left(\frac{1}{2}\right)\sinh\left(2\pi\sin\left(\frac{1}{2}\right)\right)}{\cos\left(2\pi\cos\left(\frac{1}{2}\right)\right) - \cosh\left(2\pi\sin\left(\frac{1}{2}\right)\right)}; \end{aligned}$$

$$(2.29) \qquad \sum_{k=1}^{\infty} \binom{p+k}{k}\zeta(p+k+1, a)\, z^k = \frac{(-1)^p}{p!}\left\{\psi^{(p)}(a) - \psi^{(p)}(a-z)\right\}$$

$$(p \in \;;\;\; \Re(a) > 0;\;\; |z| < |a|) \quad ;$$

$$(2.30) \qquad \sum_{k=2}^{\infty} \frac{k}{2^k}\,\zeta\left(k+1, \frac{3}{4}\right) = 8G$$

and

$$(2.31) \qquad \sum_{k=2}^{\infty} \frac{k}{2^k}\,\zeta\left(k+1, \frac{5}{4}\right) = 8(1-G),$$

where G denotes Catalan's constant given by

$$(2.32) \qquad G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cong 0.915965594177219015\cdots.$$

REMARK 2. Since the right-hand side of the relationship (1.18) [with $p!$ replaced by $\Gamma(p+1)$] is well-defined for $p \in \ \setminus \{-1\}$, the summation formula (2.29) may be put in a slightly more general form (*cf.* Wilton [28]; see also Srivastava [25, p. 137, Equation (6.6)]):

$$(2.33) \qquad \sum_{k=0}^{\infty} \binom{s+k-1}{k}\zeta(s+k, a)\, z^k = \zeta(s, a-z)$$

$$(s \in \ \setminus \{1\};\;\; a \neq 0, -1, -2, \cdots;\;\; |z| < |a|) \quad .$$

The foregoing method has been implemented in *Mathematica* (Version 3.0).

3. Series Involving Polygamma Functions

In view of the relationship (1.18) and (1.19), the foregoing techniques can be applied also to series involving Polygamma functions and generalized harmonic numbers. We first state

PROPOSITION 5. *Let p be a positive integer. Then, in terms of the function $\Phi(z,s,a)$ defined by*

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\qquad (|z|<1;\;\; a\neq 0,-1,-2,\cdots),\tag{3.1}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\psi^{(p)}(a+k)\,z^k&=\frac{(-1)^{p+1}\;p!}{a^{p+1}}+\frac{\psi^{(p)}(a+1)}{1-z}\\&\qquad+\frac{(-1)^p\;p!\;z^2}{1-z}\,\Phi(z,p+1,a+1)\end{aligned}\tag{3.2}$$

$$(|z|<1;\quad a\neq -1,-2,-3,\cdots).$$

Proof. Denoting, for convenience, the left-hand side of the summation formula (3.2) by $\Xi(z)$, it is not difficult to find from (1.18) and (2.2) that

$$\begin{aligned}\Xi(z)&=(-1)^{p+1}\int_0^{\infty}\frac{t^p\,e^{-(a-1)t}}{e^t-1}\left(\sum_{k=1}^{\infty}z^k\;e^{-kt}\right)dt\\&=-\int_0^1\frac{\tau^{a-1}(\log\tau)^p}{(1-\tau)(1-z\tau)}\,d\tau\qquad(\Re(a)>0).\end{aligned}$$

Upon evaluating this last integral, and waiving the restriction on the parameter a by appealing to the principle of analytic continuation, we complete the proof of Proposition 5.

Next we turn to a family of *linear* harmonic sums:

$$\mathcal{S}_{p,q}:=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q},\tag{3.3}$$

which were discussed extensively by Flajolet and Salvy [15]. By applying (1.19), (2.1), and (2.2), it is not difficult to prove

PROPOSITION 6. *Let $\mathcal{S}_{p,q}$ be defined by* (3.3). *Then*

$$\mathcal{S}_{p,q}=\zeta(p)\zeta(q)+\frac{(-1)^p}{(p-1)!}\int_0^1(\log t)^{p-1}\,\mathrm{Li}_q(t)\,\frac{dt}{1-t}\tag{3.4}$$

and

$$\mathcal{S}_{p,q}=\zeta(p+q)-\frac{(-1)^q}{(q-1)!}\int_0^1(\log t)^{q-1}\,\mathrm{Li}_p(t)\,\frac{dt}{1-t}.\tag{3.5}$$

REMARK 3. In view of the symmetry relation [15]:

$$\mathcal{S}_{p,q} + \mathcal{S}_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q), \tag{3.6}$$

the integral representations (3.4) and (3.5) are essentially the same. Although, in general, the integrals occurring in (3.4) and (3.5) cannot be evaluated in closed forms, many interesting particular cases of linear harmonic sums would follow from Proposition 6.

4. Series Involving Polylogarithmic Functions

In this section we shall derive, among other results, an integral representation for the Khintchine constant K_0 which arises in the measure theory of continued fractions. Every positive irrational number μ can indeed be written uniquely as a simple continued fraction:

$$\mu = \frac{a_0}{a_1+}\frac{a_2}{a_3+}\cdots\frac{a_{n-1}}{a_n+}\cdots, \tag{4.1}$$

that is, with a_0 a non-negative integer and with all other a_j $(j = 1, 2, 3, \cdots)$ positive integers. The Gauss-Kuz'min distribution (*cf.*, *e.g.*., [19]) predicts that the density of occurrence of some chosen positive integer k in the continued fraction (4.1) of a random real number is given by

$$\mathrm{Prob}\{a_n = k\} = -\log_2\left(1 - \frac{1}{(k+1)^2}\right). \tag{4.2}$$

And, making use of the Gauss-Kuz'min distribution involving (4.2), Khintchine [19] showed that, for almost all irrational numbers, the limiting geometric mean of the positive integer elements a_j $(j = 1, 2, 3, \cdots)$ of the relevant continued fraction exists and equals

$$\begin{aligned} K_0 &:= \prod_{k=1}^{\infty}\left\{1 + \frac{1}{k(k+2)}\right\}^{\log_2 k} \\ &= \prod_{k=1}^{\infty}\left\{k^{\log_2\left(1+\frac{1}{k(k+2)}\right)}\right\}. \end{aligned} \tag{4.3}$$

An interesting explicit representation of the Khintchine constant K_0 in terms of polylogarithmic functions was proven recently by Bailey *et al.* [4, p. 422]:

$$\log(K_0)\log 2 = (\log 2)^2 + \mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n\,\mathrm{Li}_2\left(\frac{4}{n^2}\right). \tag{4.4}$$

If we set

$$\mathcal{L}(n) := \sum_{n=2}^{\infty}(-1)^n\,\mathrm{Li}_2\left(\frac{4}{n^2}\right), \tag{4.5}$$

replace the polylogarithmic function by its series representation given by (1.12) with $s = 2$, change the order of summation, and evaluate the inner sum, we shall obtain

$$\mathcal{L}(n) = 2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k^2} - \sum_{k=1}^{\infty}\{\zeta(2k) - 1\}\,\frac{4^k}{k^2}. \tag{4.6}$$

It seems very unlikely that the sums occurring in (4.6) can be evaluated in terms of well-known functions. Nevertheless, by noting that

$$\sum_{k=1}^{\infty}\frac{t^k}{k^2}\,\zeta(2k) = \int_0^1\frac{dt}{t}\sum_{k=1}^{\infty}\frac{t^k}{k}\,\zeta(2k), \tag{4.7}$$

and evaluating the inner sum by the method illustrated in the preceding sections, we find that

$$\sum_{k=1}^{\infty} \frac{t^k}{k^2}\,\zeta(2k) = \log\left(\pi\sqrt{t}\,\csc(\pi\sqrt{t}\,)\right). \tag{4.8}$$

Combining (4.6), (4.7), and (4.8), we have

$$\mathcal{L}(n) := \sum_{n=2}^{\infty} (-1)^n \operatorname{Li}_2\left(\frac{4}{n^2}\right) = \int_0^1 \frac{dt}{t}\log\left(\frac{\pi\sqrt{t}\,\cot(\pi\sqrt{t}\,)}{1-4t}\right), \tag{4.9}$$

which leads us immediately to the following (presumably new) integral representation for the Khintchine constant K_0:

$$\log(K_0)\log 2 = \frac{\pi^2}{12} + \frac{(\log 2)^2}{2} + \int_0^{\pi} \log\left(t|\cot t|\right)\frac{dt}{t}. \tag{4.10}$$

Other sums involving the polylogarithmic function are given below.

$$\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^k \operatorname{Li}_k(z^2) = 1 - z^{-1}\operatorname{arctanh} z; \tag{4.11}$$

$$\sum_{k=1}^{\infty} 2^{-k}\operatorname{Li}_k(z^2) = \frac{1}{2}\,z\log\left(\frac{1+z}{1-z}\right). \tag{4.12}$$

Since $\operatorname{arctanh}\left(\frac{1}{2}\right) = \log 3$, both (4.11) and (4.12) can be expressed in terms of $\log 3$ when $z = \frac{1}{2}$.

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