Fault-Tolerant Universality from Fault-Tolerant Stabilizer Operations and Noisy Ancillas

Ben W. Reichardt

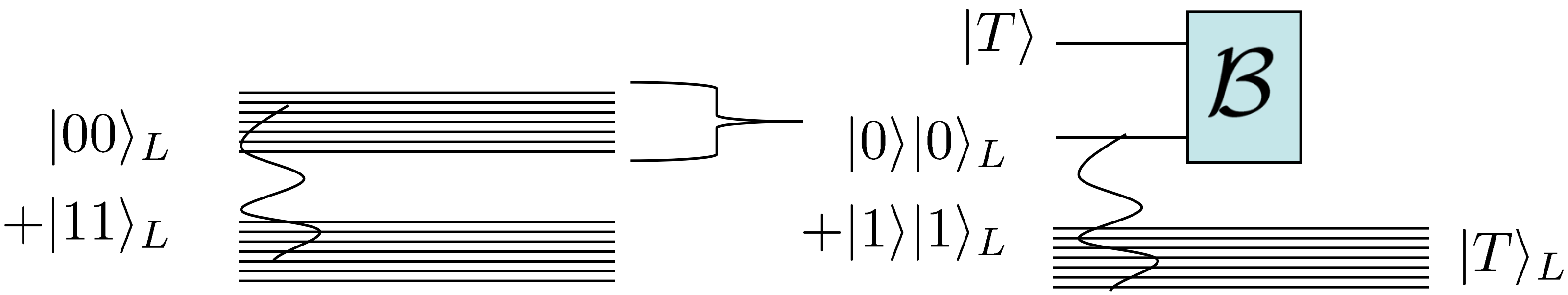
UC Berkeley

NSF, ARO

[quant-ph/0411036]

Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?

Motivation: [Knill '04] Estimated threshold of 5-10%.

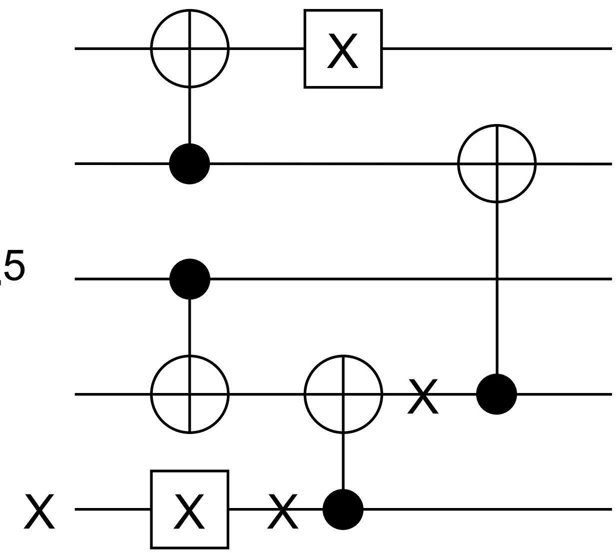


Stabilizer op. fault-tolerance $\rightsquigarrow$ Universal fault-tolerance

Stabilizer operations

Def: Stabilizer operations = CNOT, Hadamard, Phase gates,
+ Prepare, measure $|0\rangle/|1\rangle$.

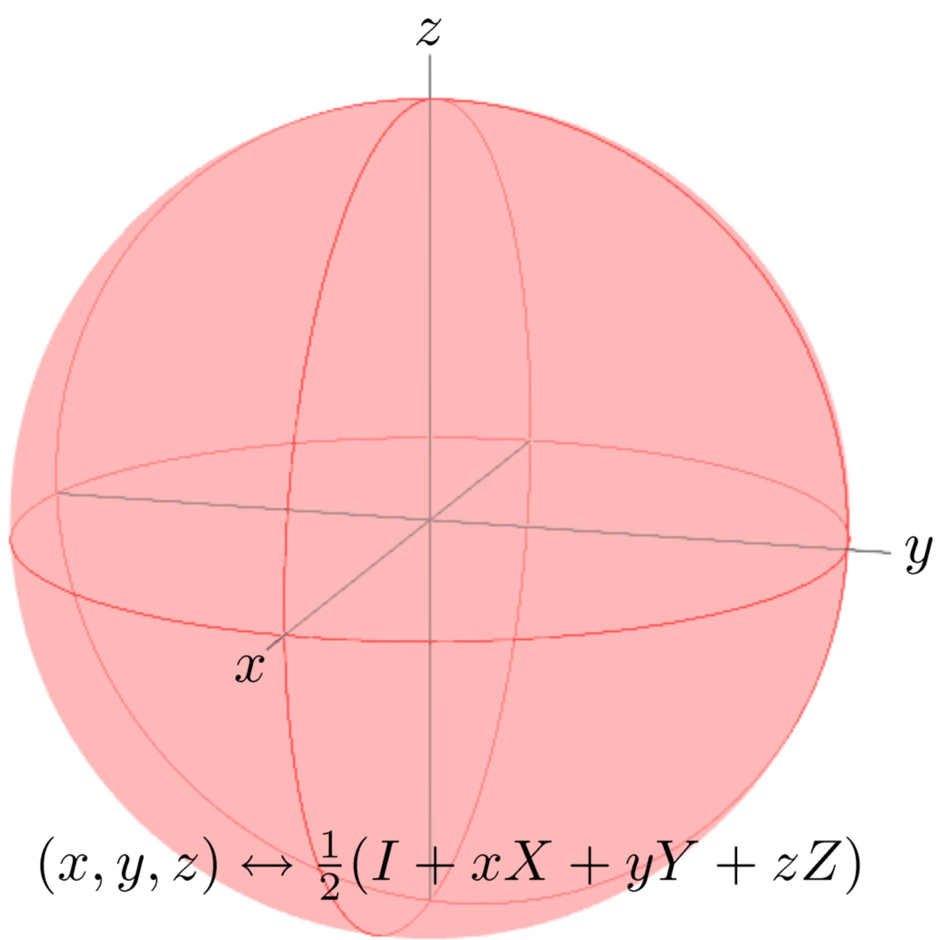
x 2 {0,1}^{5}



$$y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + x \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

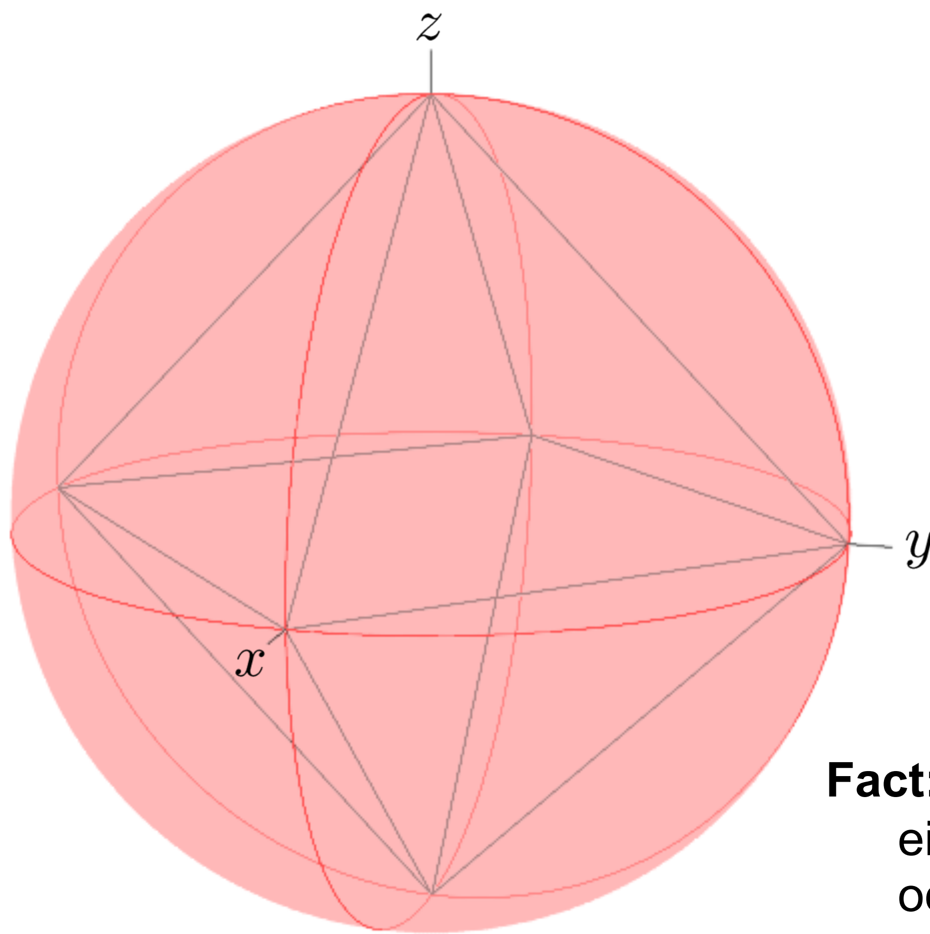
Gottesman-Knill Theorem: Stabilizer operations are
efficiently classically simulable.

Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?



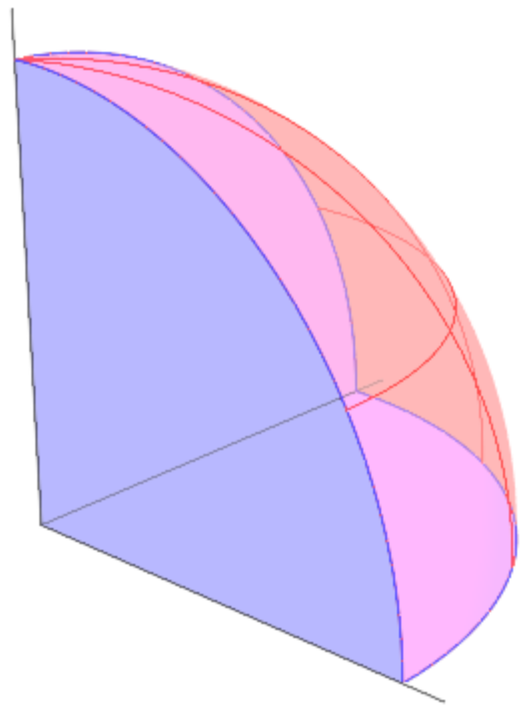
$$(x, y, z) \leftrightarrow \frac{1}{2}(I + xX + yY + zZ)$$

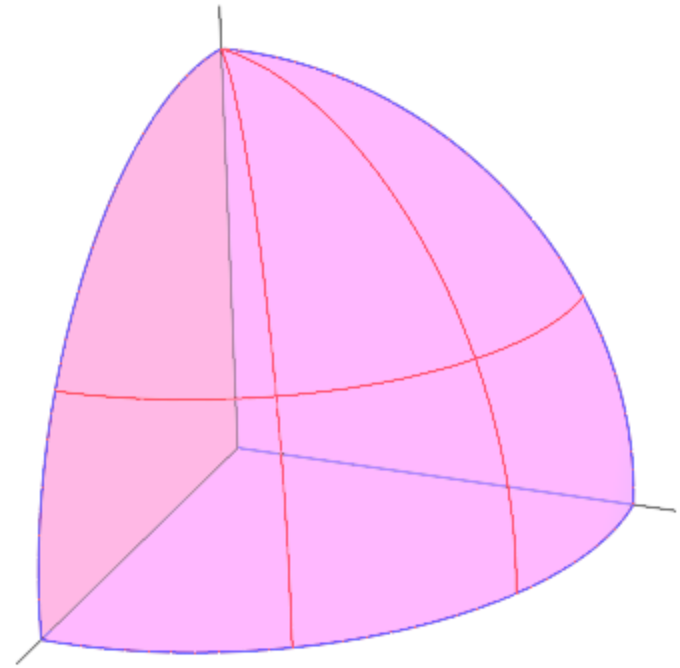
Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?



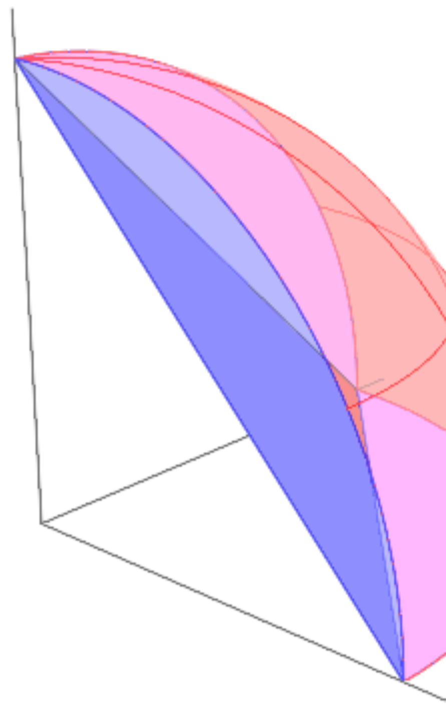
Fact: Any mixture of Pauli eigenstates (points in octahedron) is classically simulable.

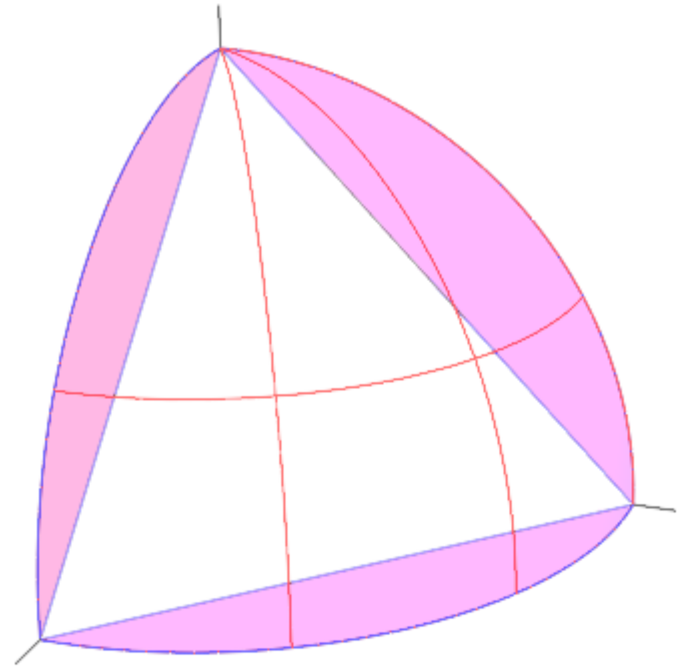
Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?





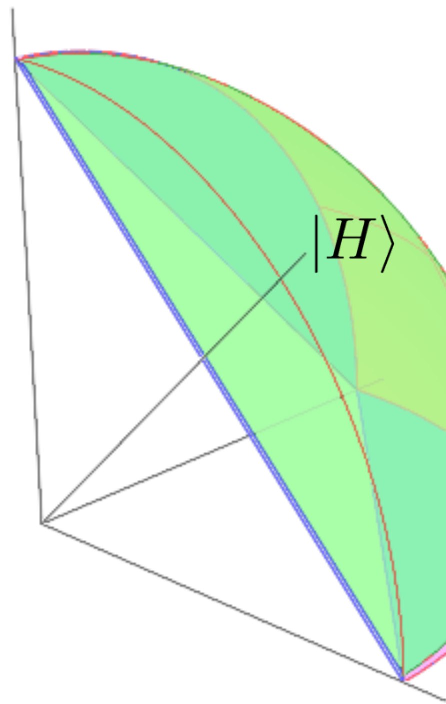
Q: Do $\left\{ \begin{array}{l} \text{stabilizer operations,} \\ \text{prepare } \rho \end{array} \right\}$ form a universal set?

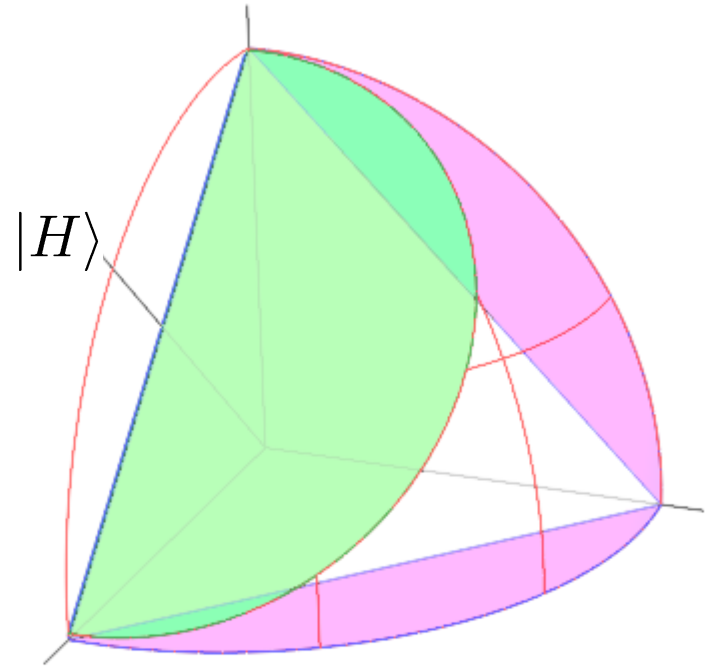




Fact: Any mixture of Pauli eigenstates (points in octahedron) is classically simulable.

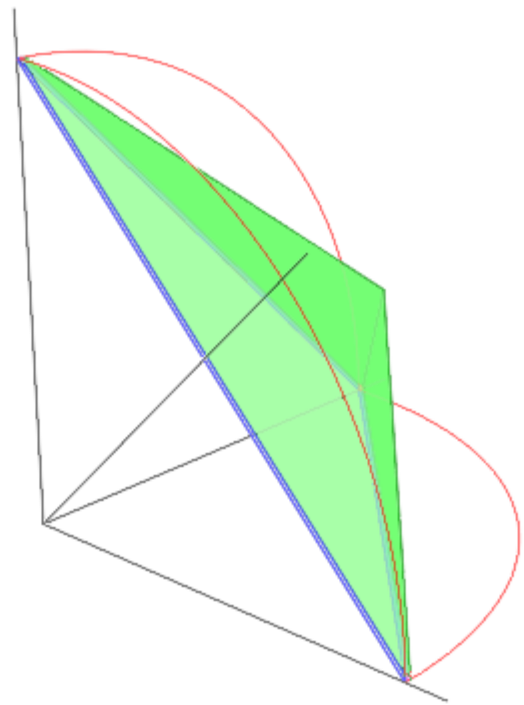
Q: Do $\left\{ \begin{array}{l} \text{stabilizer operations,} \\ \text{prepare } \rho \end{array} \right\}$ form a universal set?

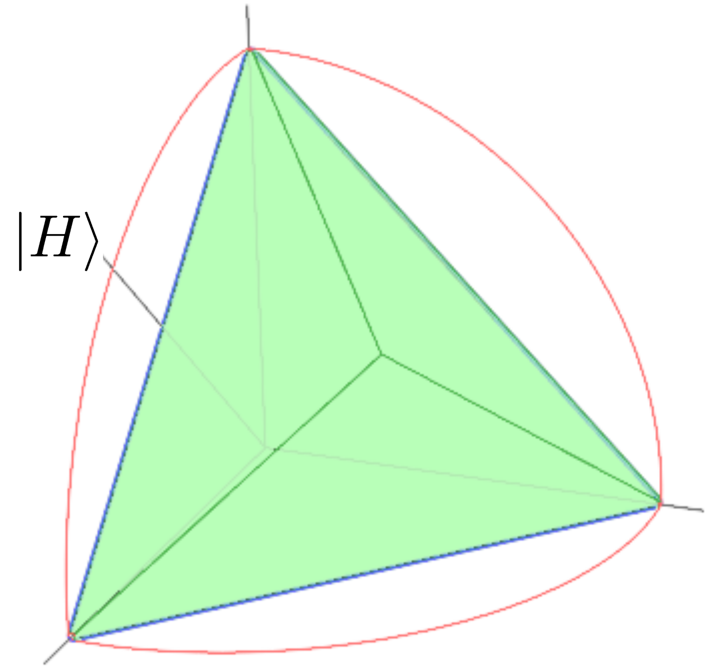




[Bravyi-Kitaev ‘04, Knill ‘04] Yes for $|H\rangle$ w/ <14.2% error

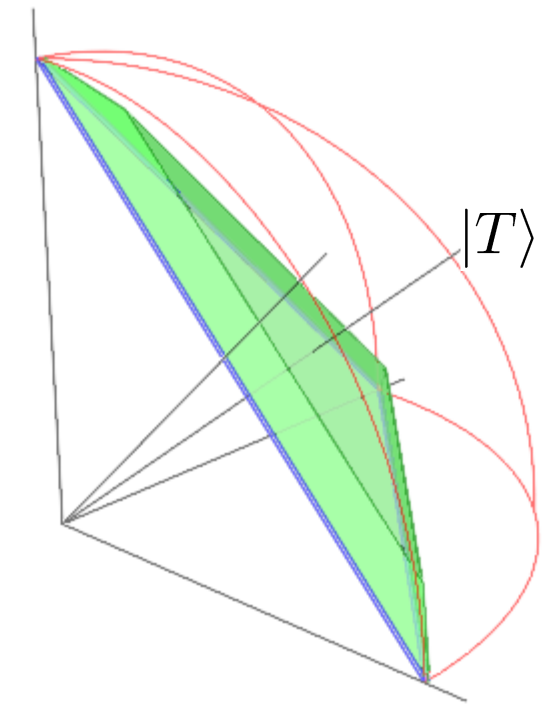
Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?

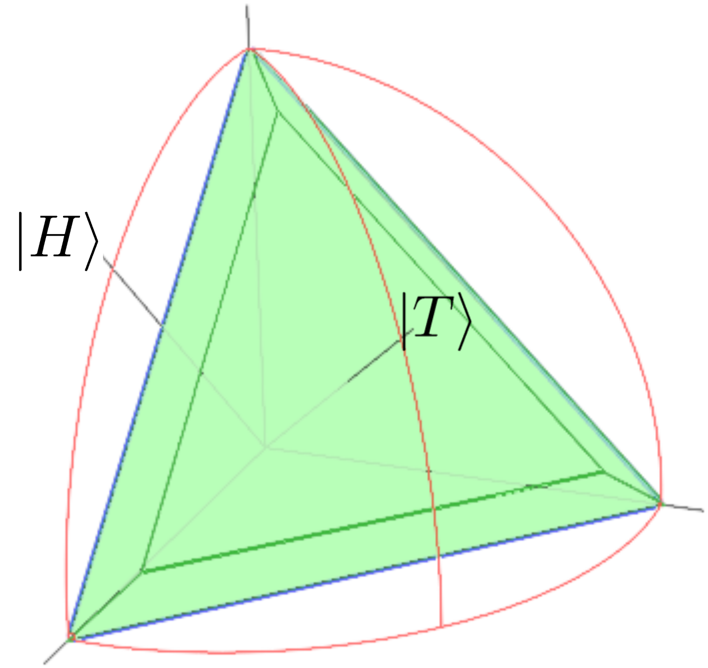




[Bravyi-Kitaev ‘04, Knill ‘04] Yes for $|H\rangle$ w/ <14.2% error

Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?



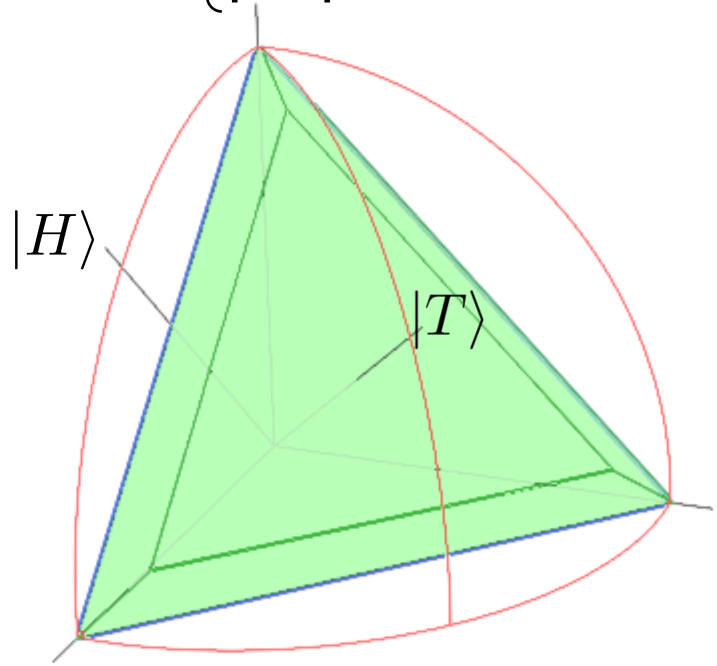


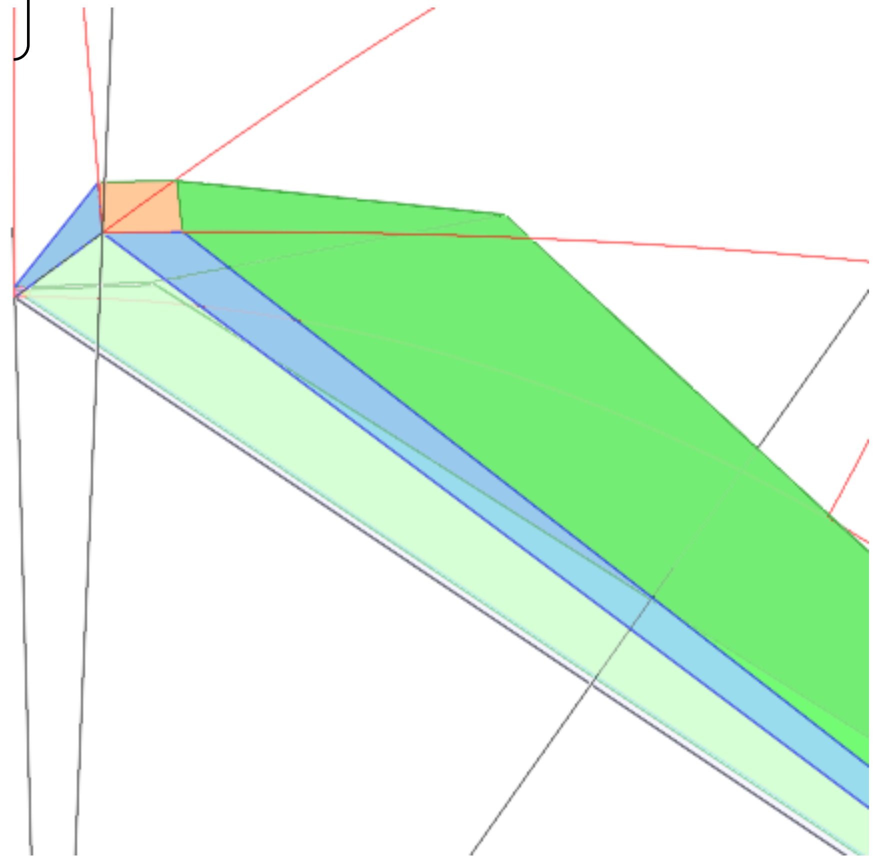
[Bravyi-Kitaev '04, Knill '04] Yes for $|H\rangle$ w/ <14.2% error

[Bravyi-Kitaev '04] Yes for $|T\rangle$ w/ <17.3% error

$= \frac{1}{2}(1 - \sqrt{\frac{3}{7}})$

Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?



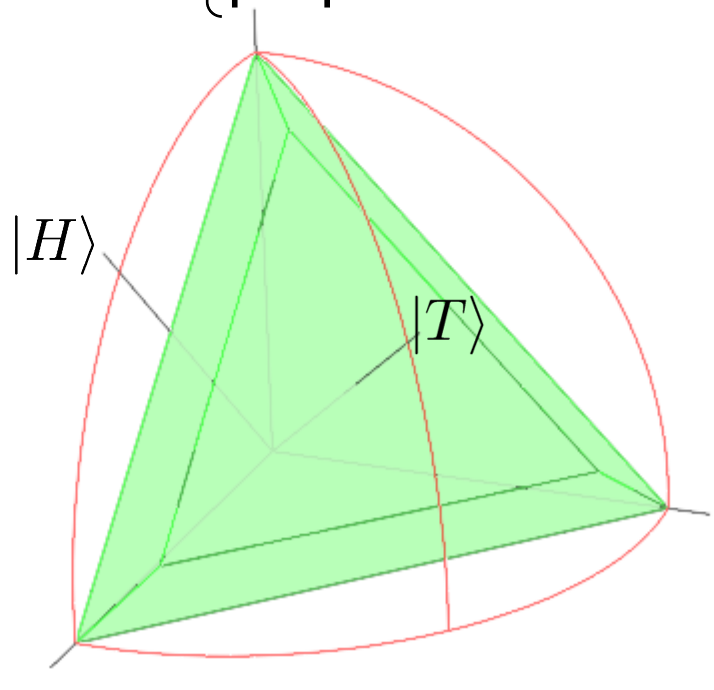


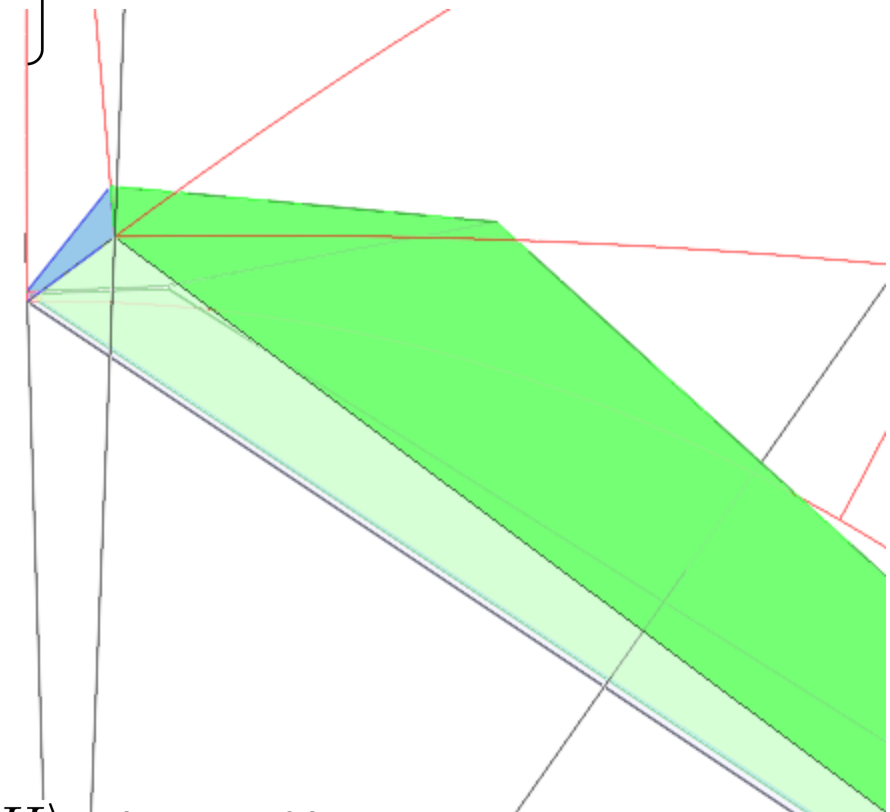
[Bravyi-Kitaev ‘04, Knill ‘04] Yes for $|H\rangle$ w/ <14.2% error

[Bravyi-Kitaev ‘04] Yes for $|T\rangle$ w/ <17.3% error

$= \frac{1}{2}(1 - \sqrt{\frac{3}{7}})$

Q: Do $\left\{ \begin{array}{l} \text{stabilizer operations,} \\ \text{prepare } \rho \end{array} \right\}$ form a universal set?



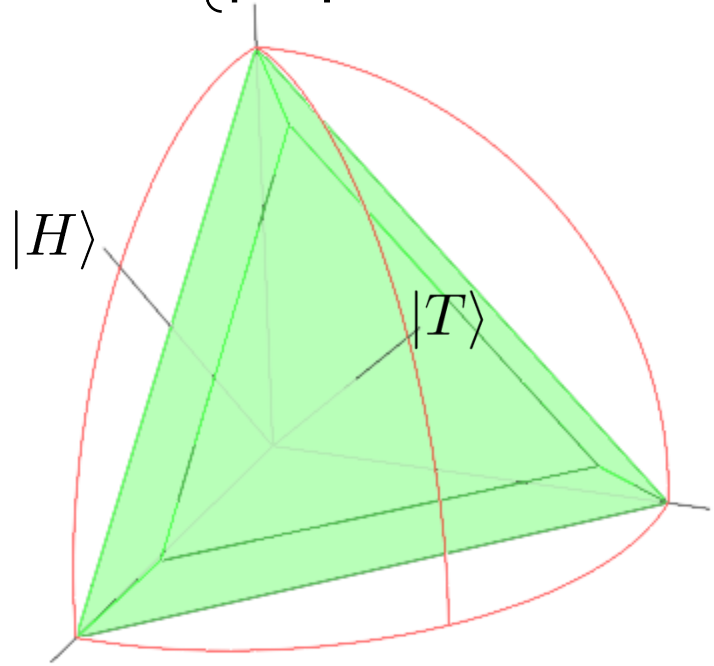


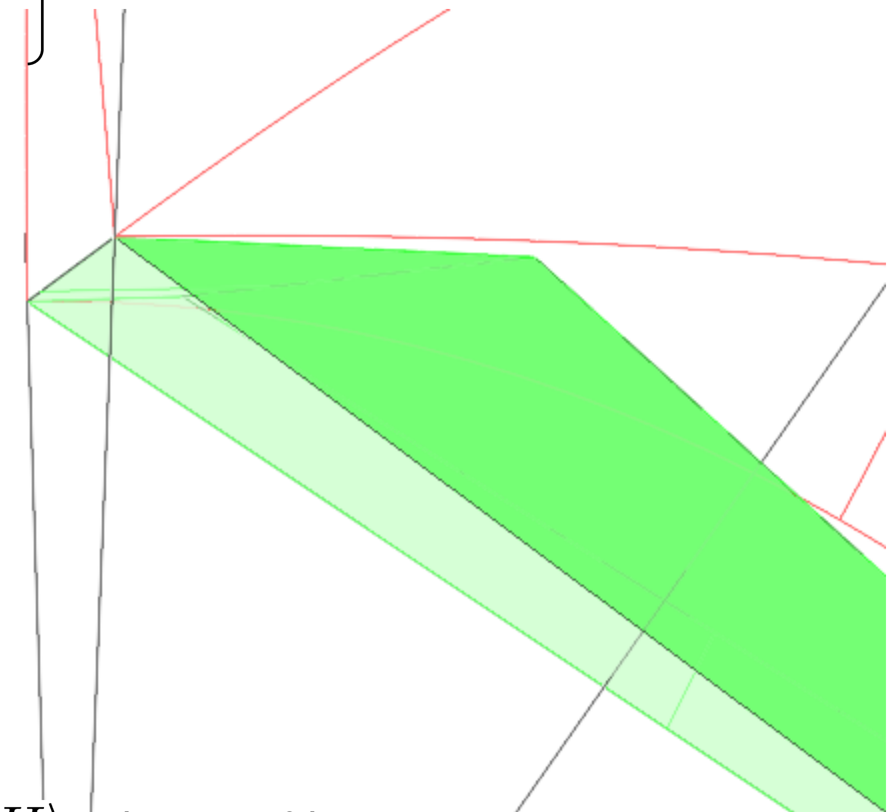
[Bravyi-Kitaev ‘04, Knill ‘04] Yes for $|H\rangle$ w/ <14.2% error

[Bravyi-Kitaev ‘04] Yes for $|T\rangle$ w/ <17.3% error $= \frac{1}{2}(1 - \sqrt{\frac{3}{7}})$

Theorem: [R ‘04] Yes for $|H\rangle$ w/ <14.6% error $= \frac{1}{2}(1 - \frac{1}{\sqrt{2}})$

Q: Do $\left\{ \begin{array}{l} \text{stabilizer operations,} \\ \text{prepare } \rho \end{array} \right\}$ form a universal set?





[Bravyi-Kitaev '04, Knill '04] Yes for $|H\rangle$ w/ <14.2% error

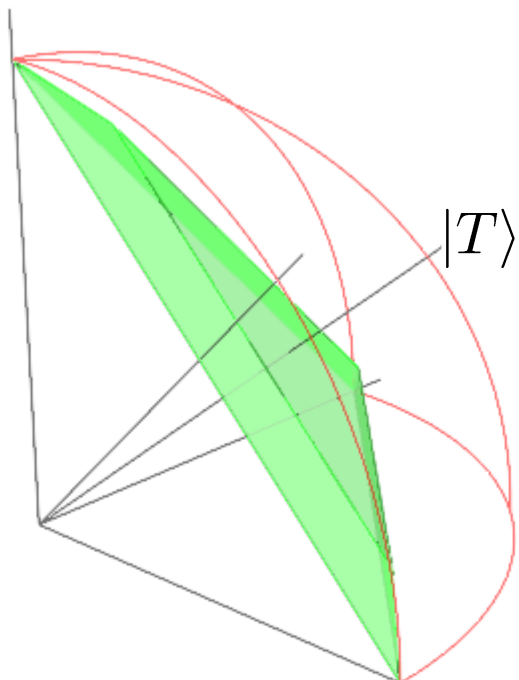
[Bravyi-Kitaev '04] Yes for $|T\rangle$ w/ <17.3% error

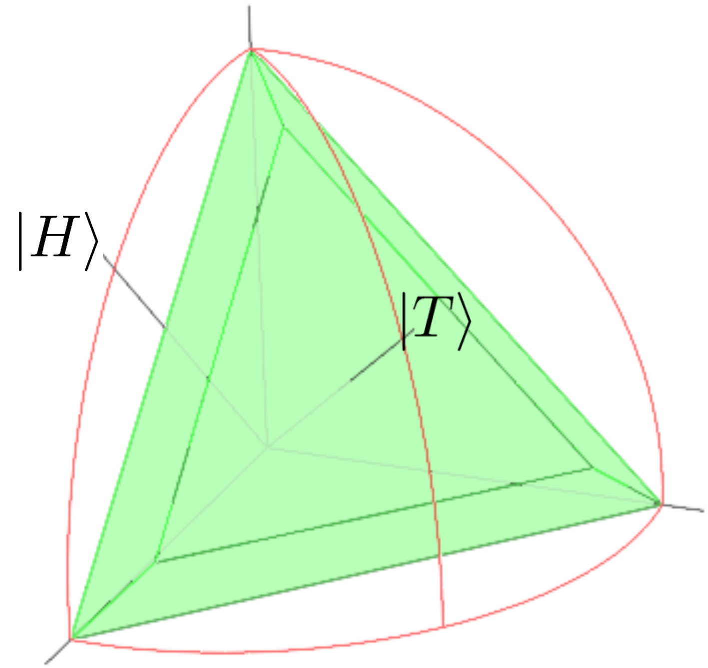
$= \frac{1}{2}(1-\sqrt{\frac{3}{7}})$

Theorem: [R '04] Yes for $|H\rangle$ w/ <14.6% error

$= \frac{1}{2}(1-\frac{1}{\sqrt{2}})$

Q: Do $\left\{\begin{array}{l}\text{stabilizer operations,}\\ \text{prepare } \rho\end{array}\right\}$ form a universal set?





[Bravyi-Kitaev '04, Knill '04] Yes for $|H\rangle$ w/ <14.2% error

[Bravyi-Kitaev '04] Yes for $|T\rangle$ w/ <17.3% error $= \frac{1}{2}(1 - \sqrt{\frac{3}{7}})$

Theorem: [R '04] Yes for $|H\rangle$ w/ <14.6% error $= \frac{1}{2}(1 - \frac{1}{\sqrt{2}})$

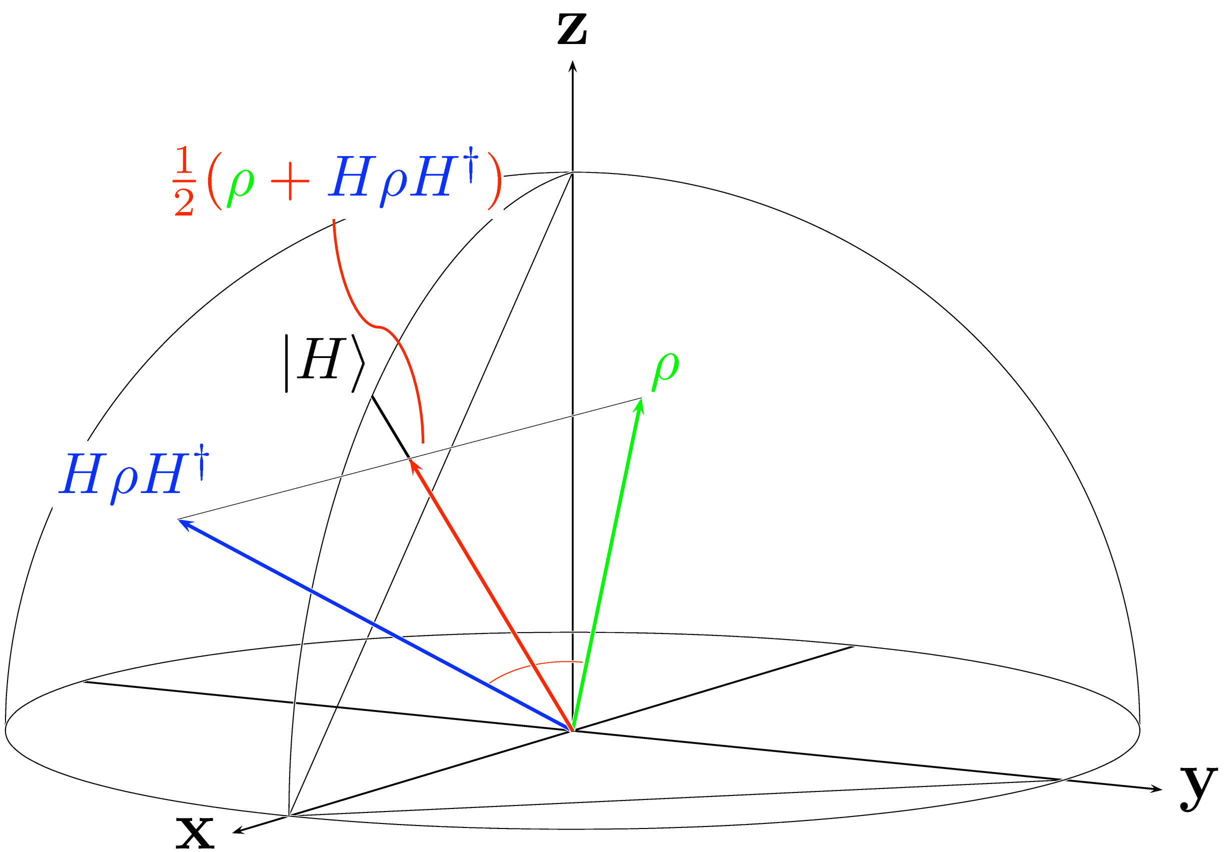
Improved distillation procedure

1. With equal probabilities ½, apply H to ρ .

) Assume ρ lies along H axis: $\rho = \frac{1}{2}(I + x(X+Z))$

$$= \frac{1}{2}\begin{pmatrix} 1+x & 1+x \\ 1+x & 1-x \end{pmatrix}$$

$x =$



$$\rho = \frac{1}{2}(I + x(X+Z)) = \frac{1}{2}\begin{pmatrix} 1+x & 1+x \\ 1+x & 1-x \end{pmatrix}$$

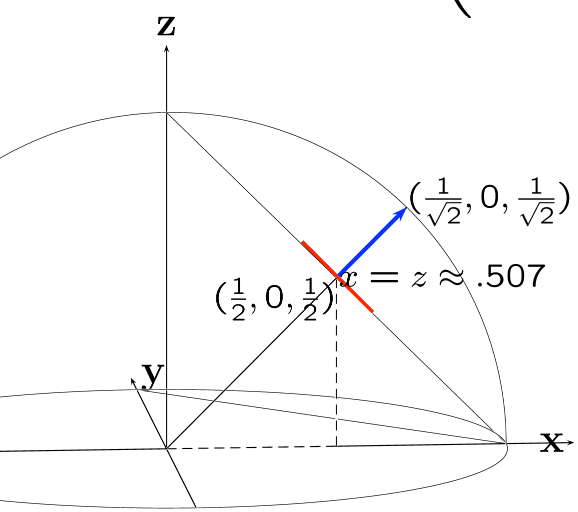
Improved distillation procedure

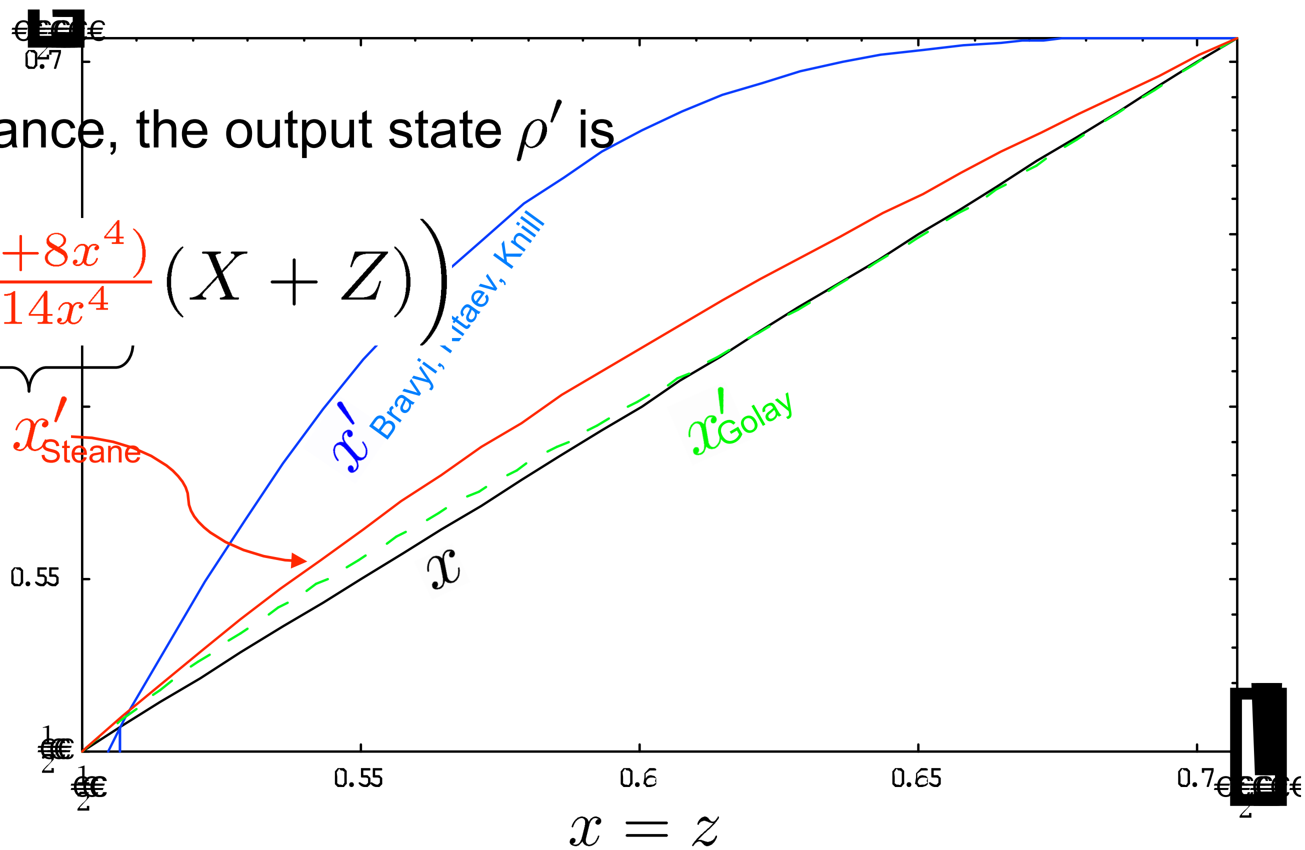
1. Symmetrize ρ into $\rho = \frac{1}{2}(I + x(X+Z)) = \frac{1}{2}\begin{pmatrix} 1+x & 1+x \\ 1+x & 1-x \end{pmatrix}$.

2. Take 7 copies of ρ. Decode according to the [[7,1,3]] Steane/Hamming quantum code, rejecting if errors detected.

3. Conditioned on acceptance, the output state ρ' is

$$\rho' = \frac{1}{2}\left(I + \underbrace{\frac{x^3(7+8x^4)}{1+14x^4}}_{x'_{\text{Steane}}}(X+Z)\right)$$





Proof of improved distillation procedure

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+x & 1+x \\ 1+x & 1-x \end{pmatrix}$$

$$\rho' = \begin{pmatrix} \langle 0_L|\rho^{\otimes n}|0_L\rangle & \langle 0_L|\rho^{\otimes n}|1_L\rangle \\ \langle 1_L|\rho^{\otimes n}|0_L\rangle & \langle 1_L|\rho^{\otimes n}|1_L\rangle \end{pmatrix} /\mathrm{tr}$$

For a CSS code in which $X_L = X^{-n}$, $Z_L = Z^{-n}$,

$$|0_L\rangle = \frac{1}{\sqrt{|C|}}\sum_{a\in C}|a\rangle \qquad |1_L\rangle = X_L|0_L\rangle$$

where C is the set of codewords for a classical code.

Thus $\langle 0_L|\rho^{\otimes n}|0_L\rangle \propto \sum_{a,b\in C}\langle a|\rho^{\otimes n}|b\rangle$.

E.g. $\langle 0001111|\rho^{\otimes 7}|0110011\rangle = (\rho_{00})^1(\rho_{01})^2(\rho_{10})^2(\rho_{11})^2$.

Generally, $$\langle a|\rho^{\otimes n}|b\rangle = \rho_{00}^{\,n-\frac{1}{2}(|a|+|b|+|a\oplus b|)}\,\rho_{01}^{\,\frac{1}{2}(-|a|+|b|+|a\oplus b|)}\,\rho_{10}^{\,\frac{1}{2}(|a|-|b|+|a\oplus b|)}\,\rho_{11}^{\,\frac{1}{2}(|a|+|b|-|a\oplus b|)}$$

$\rho = (\quad \rho_{00} \quad \rho_{01} \quad)$

Universality via **Magic states distillation**

Theorem: [R, '04] Stabilizer operations + Prepare $|H\rangle$ w/ $<\frac{1}{2}(1-\frac{1}{\sqrt{2}})$ error $\Rightarrow$ Universality.

Appl. 1: Stabilizer op. fault-tolerance

$\rightsquigarrow$ Universal fault-tolerance.

Fact: Stabilizer operations

+ Any other single-qubit unitary

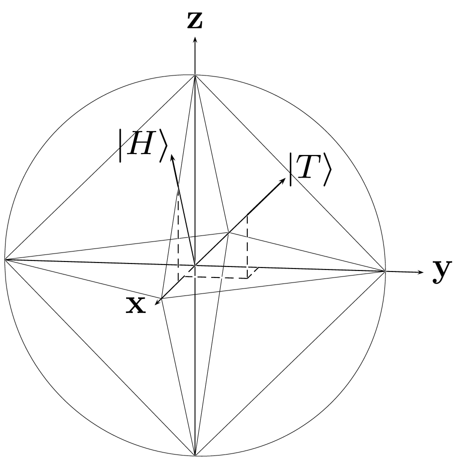
$\Rightarrow$ Universality.

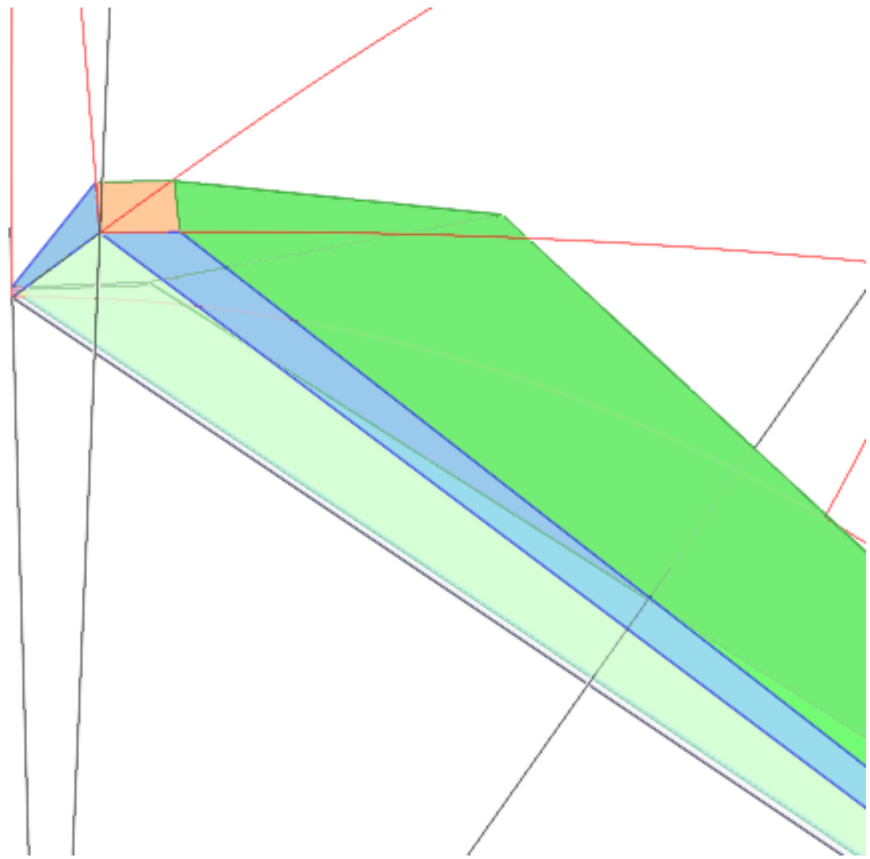
Corollary: Stabilizer operations + (ability to prepare repeatedly any pure state which is not a stabilizer state) gives universality

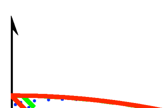
Universality from single-qubit pure states

Theorem: Stabilizer operations + (ability to prepare any single-qubit pure state which is not a Pauli eigenstate) is universal.

Proof:



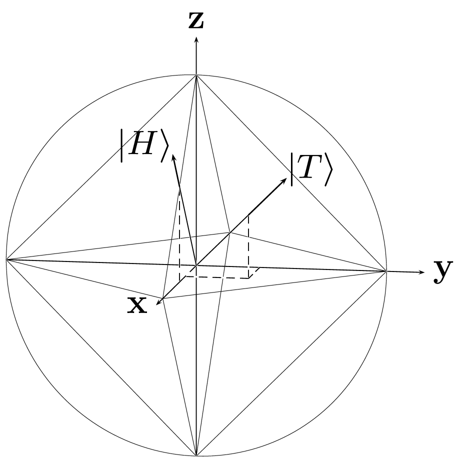


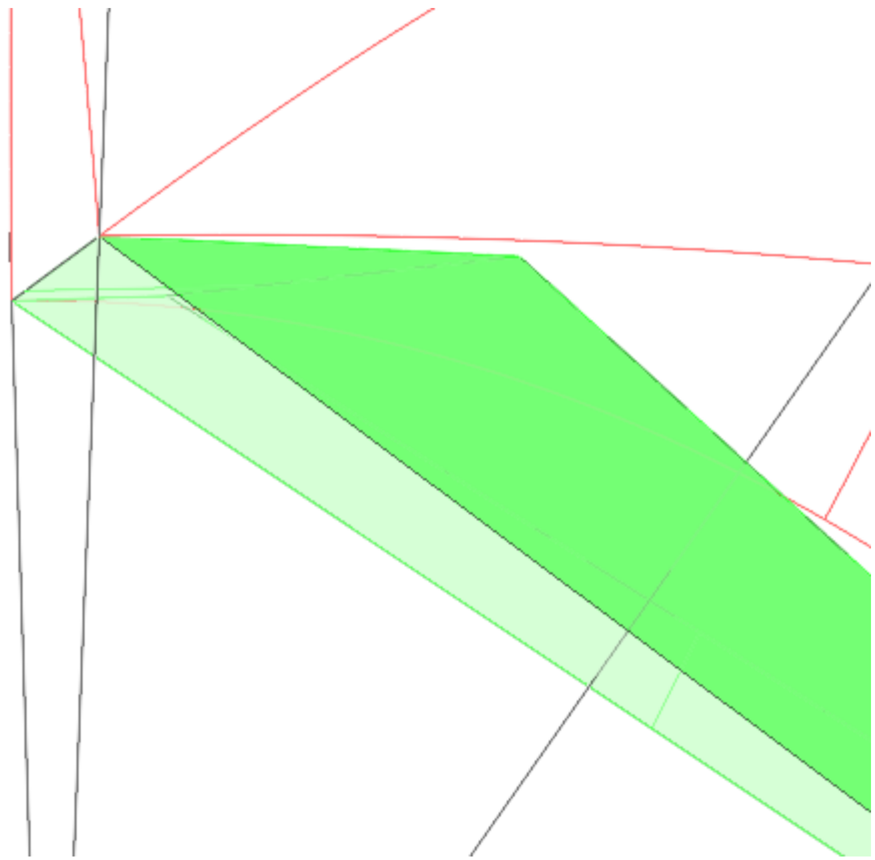


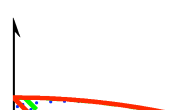
Universality from single-qubit pure states

Theorem: Stabilizer operations + (ability to prepare any single-qubit pure state which is not a Pauli eigenstate) is universal.

Proof:







Universality from **multi**-qubit pure states

Theorem: Stabilizer operations + (ability to prepare any pure state which is not a stabilizer state) is universal.

Proof: 9 sequence of Clifford unitaries and postselected Pauli measurements which reduces $|\psi\rangle$ down to a single-qubit pure state which is not a Pauli eigenstate.

By induction, true for n=1.

$$|\psi\rangle = \alpha|0\rangle|\psi_0\rangle + \beta|1\rangle|\psi_1\rangle$$

with α, $\beta \neq 0$, $|\psi_0\rangle$and $|\psi_1\rangle$ stabilizer states (else apply induction).

By applying Clifford unitaries, w.l.o.g. $|\psi_0\rangle = |0^{n-1}\rangle$.

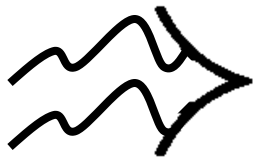
$$\cdots\ \cdots\ \cdots\ |\psi\rangle = \alpha|0\rangle|0^{n-1}\rangle + \beta|1\rangle|+^{n-1}\rangle$$

But $\alpha|0\rangle + \frac{\beta}{2^{(n-1)/2}}|1\rangle$, $\frac{\alpha}{2^{(n-1)/2}}|0\rangle + \beta|1\rangle$ can't both be stabilizer states!

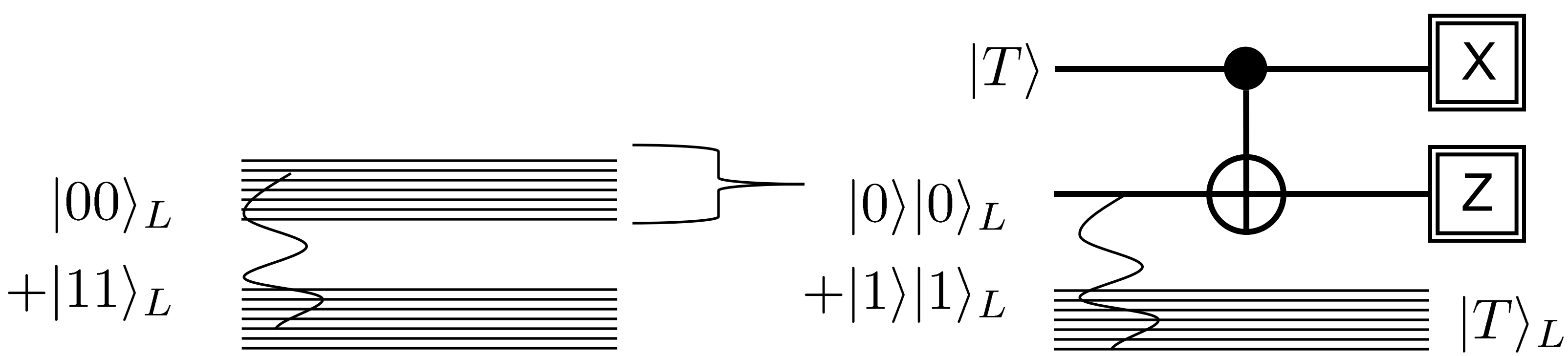
Application to fault-tolerant computing

[Knill, quant-ph/0404104]

Given scheme for fault-tolerantly applying stabilizer circuits, extend it to a universal fault-tolerant scheme.

Universal fault-tolerance  Stabilizer op. fault-tolerance

E.g., Knill's scheme has threshold of 5-10% for fault-tolerant stabilizer operations, and the same threshold for fault-tolerant universal operations.



Open questions

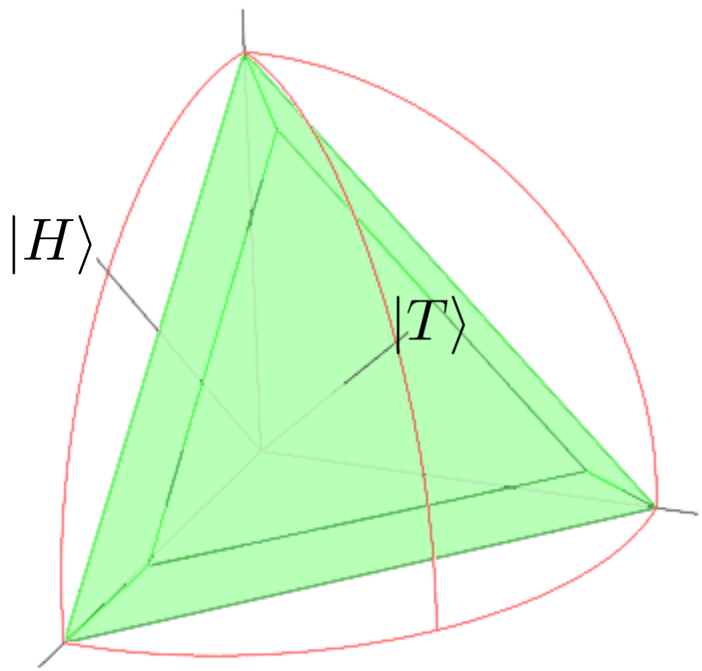
Fact: Any mixture of Pauli eigenstates (points in octahedron) is classically simulable. $\Rightarrow$ Universality from $|H\rangle$ w/ $<\frac{1}{2}(1-\frac{1}{\sqrt{2}})$ error is tight.

Open: Is stabilizer operations + (ability to prepare repeatedly single-qubit mixed state ρ) universal for all ρ outside the octahedron?

ρ

ρ

$\longrightarrow \mathbf{y}$



Open questions

ρ

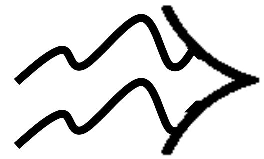
ρ

Open: Is stabilizer operations + (ability to prepare repeatedly single-qubit mixed state ρ) universal for all ρ outside the octahedron?

Open: What about perturbations to the states ρ? What about asymmetries? What if we only have fidelity lower bound? Can we characterize stable fixed points for stabilizer codes?

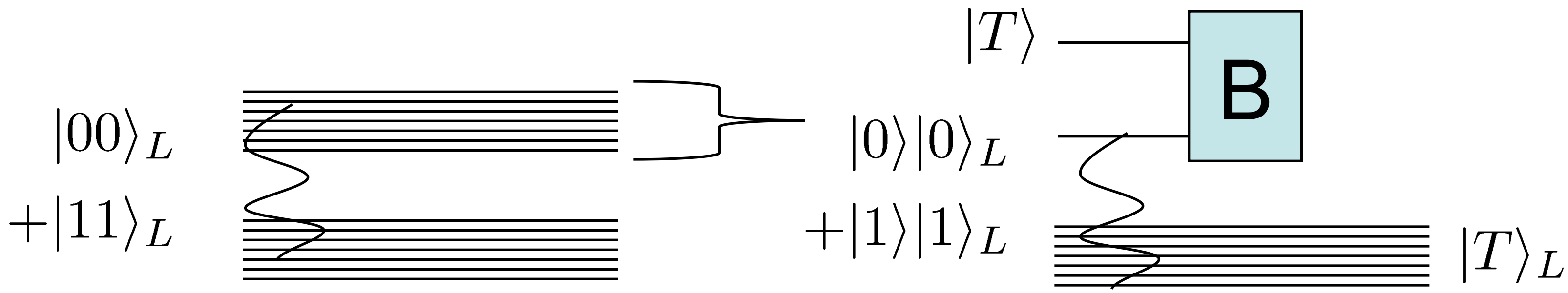
Open: Can we give a provable reduction of fault-tolerance to problem of preparing stabilizer states with independent errors?

Stabilizer op.
fault-tolerance



Universal
fault-tolerance

[Knill ‘04]



Magic states distillation:

Stabilizer operations

+ Prepare $|H\rangle$ or $|T\rangle$ with <14.6% or <17.3% error resp.

($14.6\% = \frac{1}{2}(1-\frac{1}{\sqrt{2}})$, $17.3\% = \frac{1}{2}(1-\sqrt{\frac{3}{7}})$)

[R ‘04] [Bravyi Kitaev ‘04]

$\Rightarrow$ Universality.

E.g., Knill’s scheme has same threshold for fault-tolerant stabilizer

Universality

Stabilizer operations (Clifford unitaries + prepare/measure Paulis) are not quantum universal.

Q: What additional operations are needed to get universality?

Fact: [Y. Shi, 2002] Stabilizer operations + any single-qubit unitary not in $\mathcal{C}$ is universal.

Theorem: Stabilizer operations + (ability to prepare repeatedly any pure state which is not a stabilizer state) gives universality.

Q: Is stabilizer operations + (ability to prepare single-qubit mixed state ρ) universal?