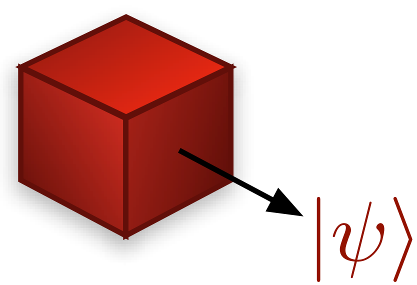
Universality by distillation

Ben W. Reichardt
Caltech

Magic states distillation problem:

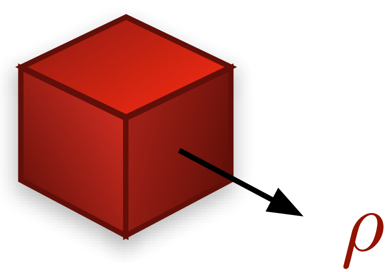
- What is the power of ancilla-assisted, classically-controlled stabilizer operations ?



(ancilla of constant size)

Magic states distillation problem:

- What is the power of noisy *ancilla*-assisted, classically-controlled stabilizer operations ?



Magic states distillation problem:

- What is the power of noisy *ancilla*-assisted, classically-controlled stabilizer operations ?

- Quantum information

Magic states distillation problem:

- What is the power of noisy *ancilla*-assisted, classically-controlled stabilizer operations?

- Quantum information
- Applications to fault tolerance:
  - FT threshold upper bounds based on Gottesman-Knill Thm
    - E.g., 15% general, 30% dephasing noise on π/8 gate [Virmani/Huelga/Plenio '05],
    - 45% depolarizing noise [Buhrman/Cleve/Laurent/Linden/Schrijver/Unger QIP 2006]
  - FT threshold lower bounds
    - E.g., estimated 3% depolarizing noise, using error detection [Knill '05],
    - proven 0.1% depolarizing noise for error-detection-based scheme [R'06]

Stabilizer operations

- Stabilizer operations = Clifford group unitaries (e.g., CNOT, H, …), Pauli operator measurement and eigenstate preparation.

- Stabilizer states = states preparable with stabilizer operations, or mixtures thereof.

- Remarks:
  - For today, stabilizer operations are perfect, not faulty.
  - Gottesman-Knill theorem efficiently simulates classically-controlled stabilizer operations.

Ancilla-assisted stabilizer operations

- Shor '97: classically-controlled stabilizer operations together with repeated preparation of

$$\frac{1}{2}\left(|000\rangle + |010\rangle + |100\rangle + |111\rangle\right)$$

gives quantum universality. (Generalized by Gottesman, Chuang '99.)

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gives quantum universality. (Generalized by Gottesman, Chuang '99.)

- Knill/Laflamme/Zurek '97: Reduced universality to preparation of

$$|H\rangle = |\pi/8\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$$

and gave a purification network for noisy ancilla states.

Ancilla-assisted stabilizer operations

- Shor '97: classically-controlled stabilizer operations and repeated preparation of $\frac{1}{2}\left(|000\rangle + |010\rangle + |100\rangle + |111\rangle\right)$ gives universality.
- Knill/Laflamme/Zurek '97: Reduced to $|H\rangle = |\pi/8\rangle$, gave a purification procedure for noisy ancilla states.
- Dennis '01: Reduced to preparation of $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle\right)$
  - New idea: Considered *recursive* purification/distillation of noisy ancilla states, showed that up to ~7% $|11\rangle$ noise can be tolerated

(In fact, up to exactly 25% of this noise can be tolerated, or exactly 40% depolarizing noise — and both numbers are tight.)

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- Dennis '01: Recursive purification of $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle\right)$

★ Bravyi/Kitaev '05: Formalized general problem, introduced recursive purification based on codes, and gave new purification protocol.

Example: Parity checks

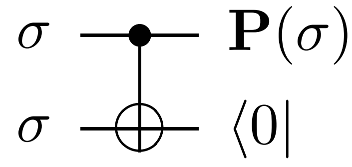
- Flip a coin with bias ϵ. To amplify the bias to ~2ϵ, flip two coins and condition on their outcomes being the same.
  - $\begin{pmatrix} \frac{1}{2}+\epsilon & 0 \\ 0 & \frac{1}{2}-\epsilon \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2}+\frac{2\epsilon}{1+4\epsilon^2} & 0 \\ 0 & \frac{1}{2}-\frac{2\epsilon}{1+4\epsilon^2} \end{pmatrix} \rightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$
  - $\rho(x,y,z) = \frac{1}{2}(I + xX + yY + zZ) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$

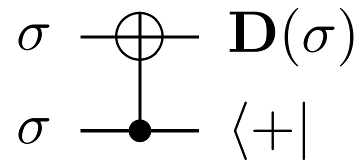
    provided z>0.

- Similarly, running parity checks in the dual basis will converge to $\rho(1,0,0) = |+\rangle\langle +|$ provided x>0.

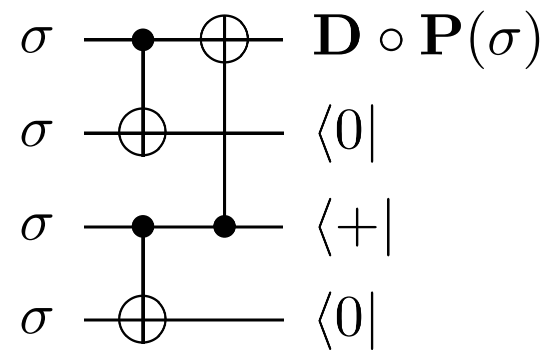
Example distillation algorithm

- **Parity** checks converge to $\rho(0,0,1) = |0\rangle\langle 0|$ if z>0.
- **Dual** p. checks converge to $\rho(1,0,0) = |+\rangle\langle +|$ if x>0.





- Repeat:
  - With probability 1/2, apply **P** then **D**
  - With probability 1/2, apply **D** then **P**

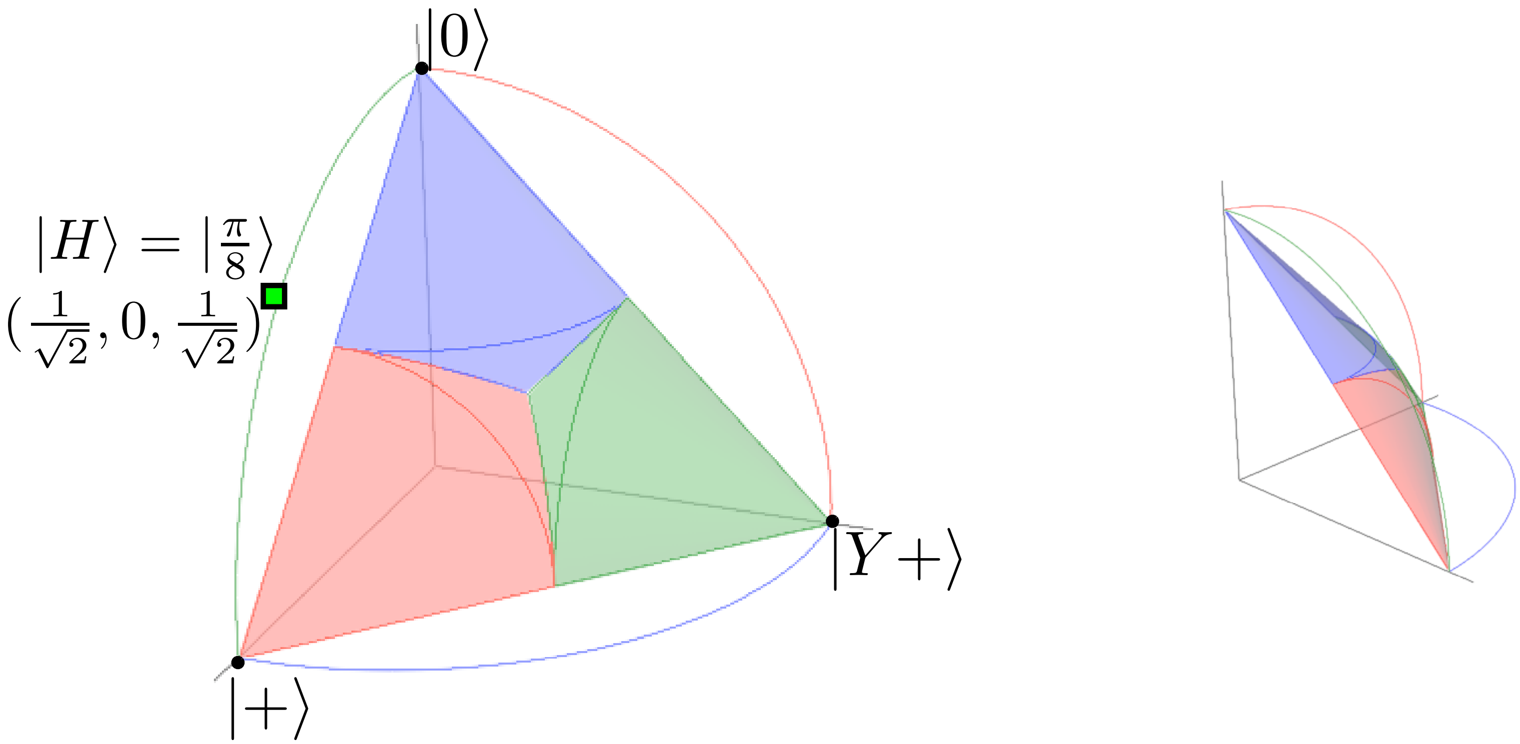


- Converges close to $\rho(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) = |H\rangle\langle H|$ Hadamard eigenstate

  provided $\rho_0 = \rho(x, 0, z)$ with x+z>1.
- Tight! If x+z=1, then ρ_0 is a mixture of $|0\rangle\langle 0|$ and $|+\rangle\langle +|$.

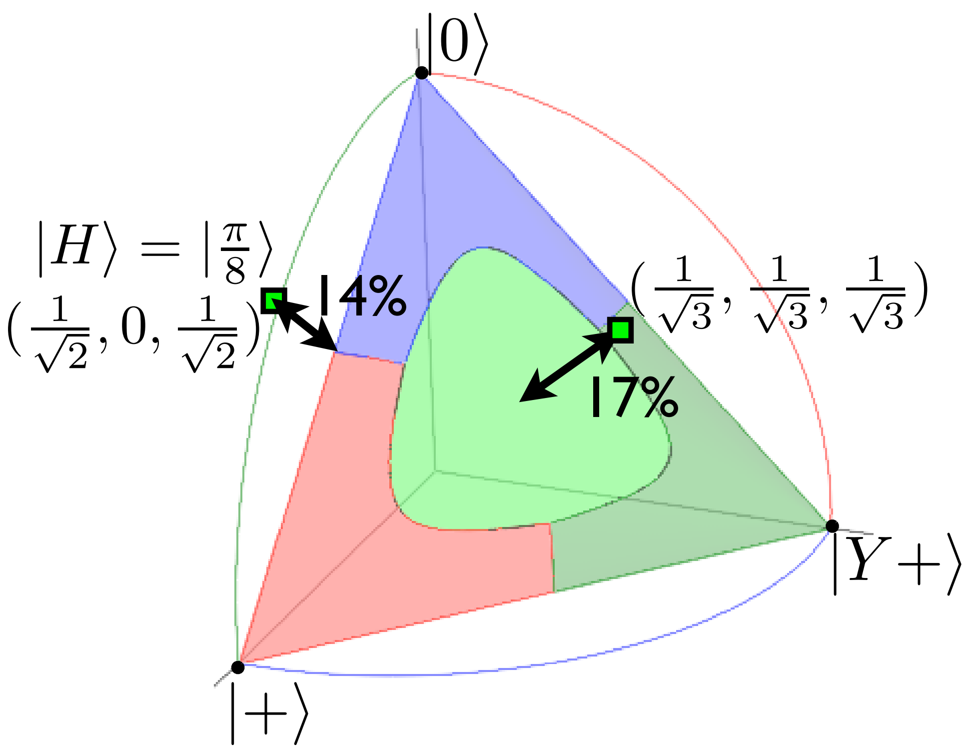
Single-qubit state-of-the-art

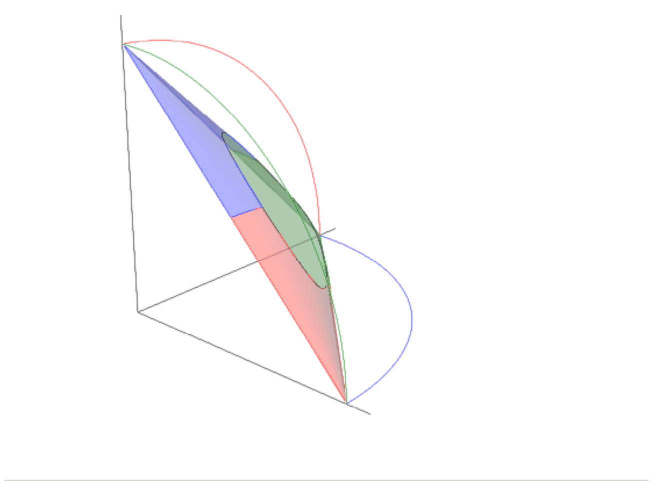
- Above **P/D** algorithm (+ more tricks) shows distillable the region beyond:



Single-qubit state-of-the-art

- Running **P** then **D** is equivalent to taking four copies of ρ, and postselecting on lying in the codespace of the four-qubit code.
- Bravyi & Kitaev used the five-qubit code to cut off $x + y + z > \frac{3}{\sqrt{7}}$

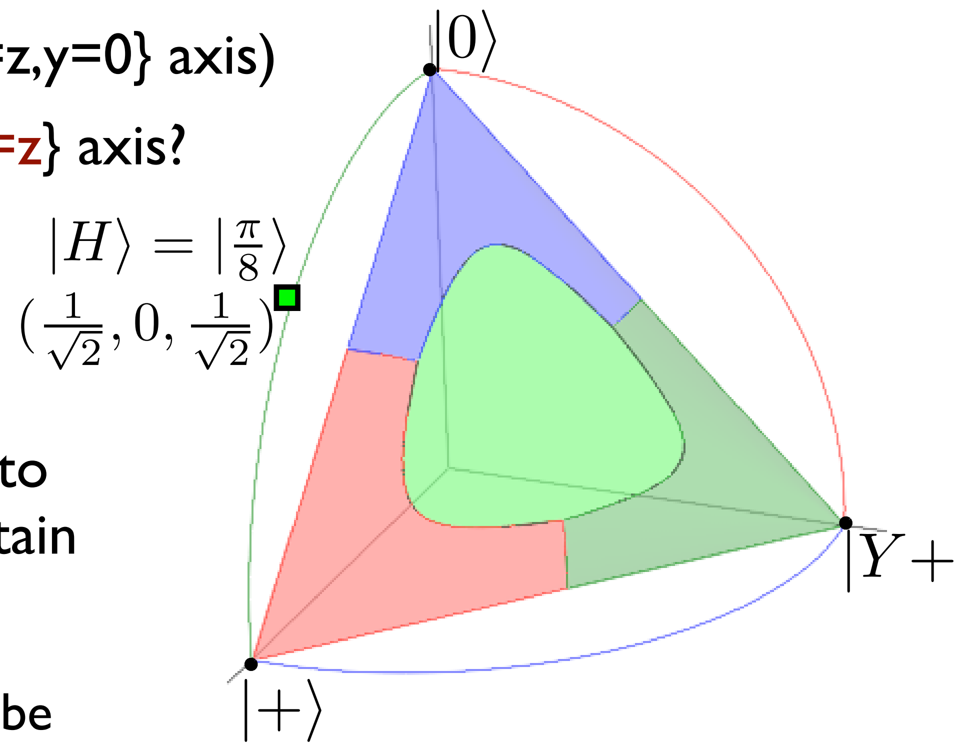




(In fact, can further round the middle corners off slightly.)

Single-qubit state-of-the-art

- Distillation is tight in H direction ({x=z,y=0} axis)
- **Open**: Can we do better along {x=y=z} axis?



- Better distillation procedure is equiv. to existence of stabilizer codes with certain weight distributions.
  - Indeed, w.l.o.g., all measurements may be assumed to have postselected outcomes
  - And no extra working space is required. □

Application: FT threshold lower bounds

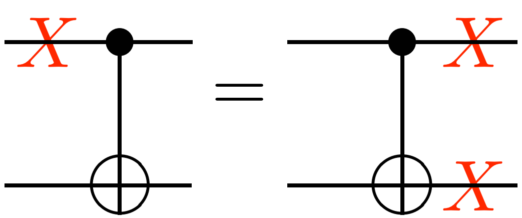
- Fault-tolerance schemes based on concatenated coding
  - Implement FT stabilizer operations at coding level k in terms of FT stabilizer operations at level k-1, …
  - But this is insufficient for universality!
- Shor implemented Toffoli_k [via preparation of level-k-encoded $\frac{1}{2}\left(|000\rangle + |010\rangle + |100\rangle + |111\rangle\right)$] in terms of Toffoli_{k-1} and stabilizers_{k-1}, …
- Alternatively, we can *teleport* a noisy ancilla state into the level-k encoding directly, then purify it with stabilizers_k.

➡ Stabilizers_k and ancilla_0 give Toffoli_k

Application: FT threshold lower bounds

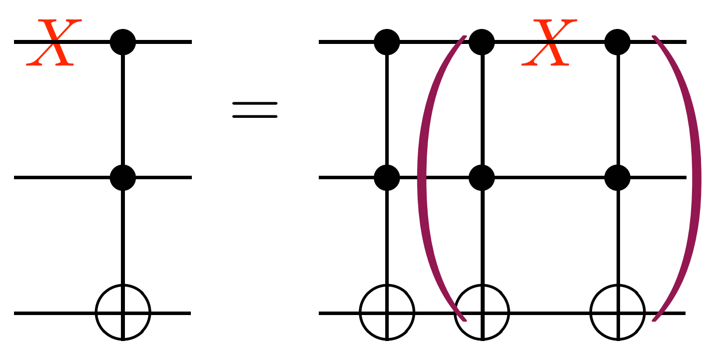
- We can *teleport* a noisy ancilla state into the level-k encoding directly, then purify it with stabilizers_k.
  - ➡ Stabilizers_k and ancilla_0 give Toffoli_k
- Advantages:
  - Magic states distillation tolerates high noise ⇒ the bottleneck is in the threshold for stabilizer operations. (Reduction)
  - Ease of analysis & simulation for discrete Pauli error models

Pauli errors pass through Cliffords:



But not past Toffolis:

Clifford!



Application: FT threshold lower bounds

- We can *teleport* a noisy ancilla state into the level-k encoding directly, then purify it with stabilizers_k.
- Results using this technique:
  - Knill '05: Estimated >3% depolarizing noise tolerable using an error-detection-based fault-tolerance scheme.
  - R. '06: Proved 0.1% noise tolerable for similar scheme, or 1.1% if noise model is known.

Practical considerations for threshold lower bounds

- Recall **P**/**D** algorithm: w/prob. 1/2, apply **P** then **D**, w/prob. 1/2, **D** then **P**.
- But in postselection/error-detection-based FT schemes, stabilizer operations can't be applied at random! (After conditioning on acceptance, coin flip will not be fair.)
- Require stability to perturbations (noise on ancilla state varies).

Theorem. There exists an $\epsilon > 0$ such that perfect CNOT, H, preparation of $|0\rangle$ and measurement in the $|0\rangle/|1\rangle$ basis, with adaptive classical control, together with the ability to prepare (unknown) states ρ_i each with fidelity $\geq 1 - \epsilon$ with $\rho(\frac{1}{\sqrt{3}}(1,1,1))$, allows efficient simulation of universal quantum computation.

Explicitly, with (x_i, y_i, z_i) the Pauli coordinates of ρ_i, $|H\rangle$ can be efficiently distilled provided $\max_i \max\{|\frac{1}{\sqrt{3}} - x_i|, |\frac{1}{\sqrt{3}} - y_i|, |\frac{1}{\sqrt{3}} - z_i|\} \leq 0.0527$.

Application: FT threshold lower bounds

- We can *teleport* a noisy ancilla state into the level-k encoding directly, then purify it with $stabilizers_k$.
- Results using this technique
  - [Knill '05]: Estimated >3% depolarizing noise tolerable using an error-detection-based fault-tolerance scheme.
  - [R. FOCS'06]: Proved 0.1% noise tolerable for similar scheme, or 1.1% if noise model is known.
- Conclusion: Lower bounds on distillable region (possibly in a more restricted model) help give lower bounds for fault-tolerance threshold.
- Open problems: Better *stable* distillation lower bounds, stable H distillation?

Application 2: FT threshold upper bounds

- Claim: Given perfect stabilizer op's,

  $\mathcal{E}$ gives universality iff $(I \otimes \mathcal{E})|\Psi\rangle$ is distillable to $|H\rangle$

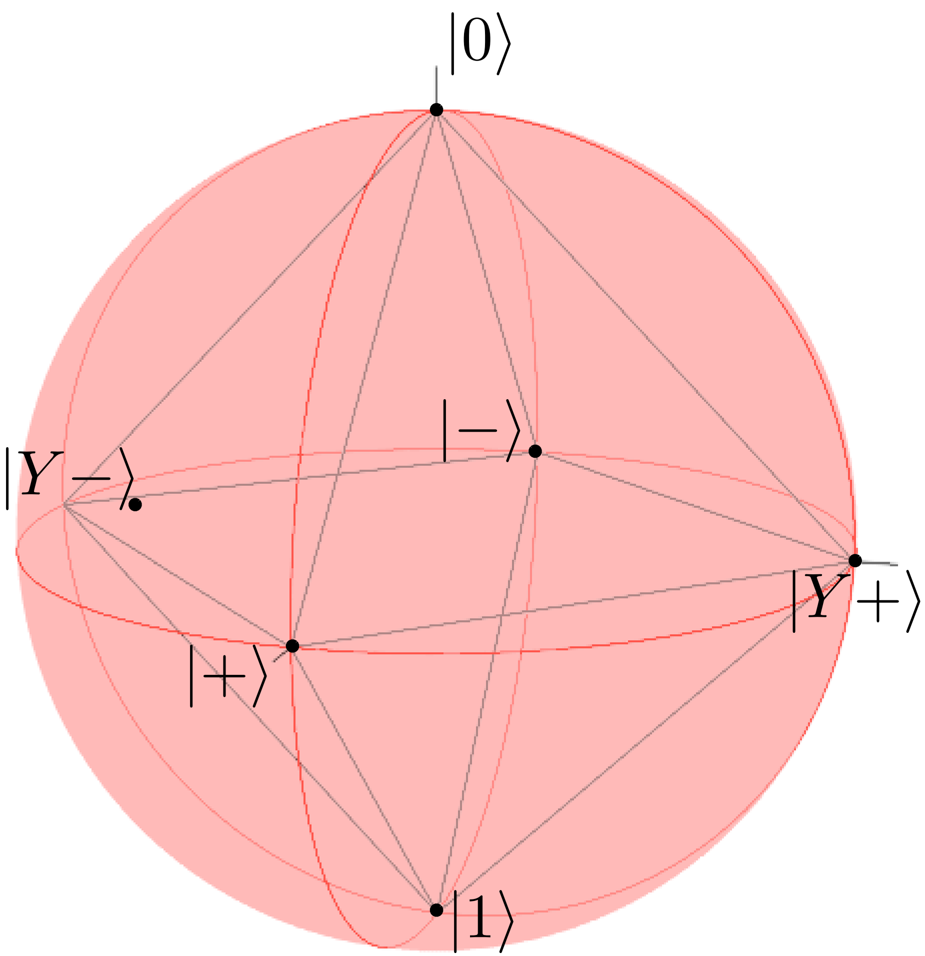
- Let $\mathcal{E}$ be noisy U. Upper bounds on distillability of $(I \otimes \mathcal{E})|\Psi\rangle$ therefore upper-bound noise on U before universality is lost.

- Distillation upper bounds: If ρ is a mixture of stabilizer states, it is not distillable to a non-stabilizer state.

(However, magic states distillation is a broader problem; not all ancillas arise from J. isom. on noisy $\mathcal{E}$).

Application 2: FT threshold upper bounds

- Let $\mathcal{E}$ be noisy U. Upper bounds on distillability of $(I \otimes \mathcal{E})|\Psi\rangle$ therefore upper-bound noise on U before universality is lost.
- Approach of [Buhrman/Cleve/Laurent/Linden/Schrijver/Unger QIP 2006]:



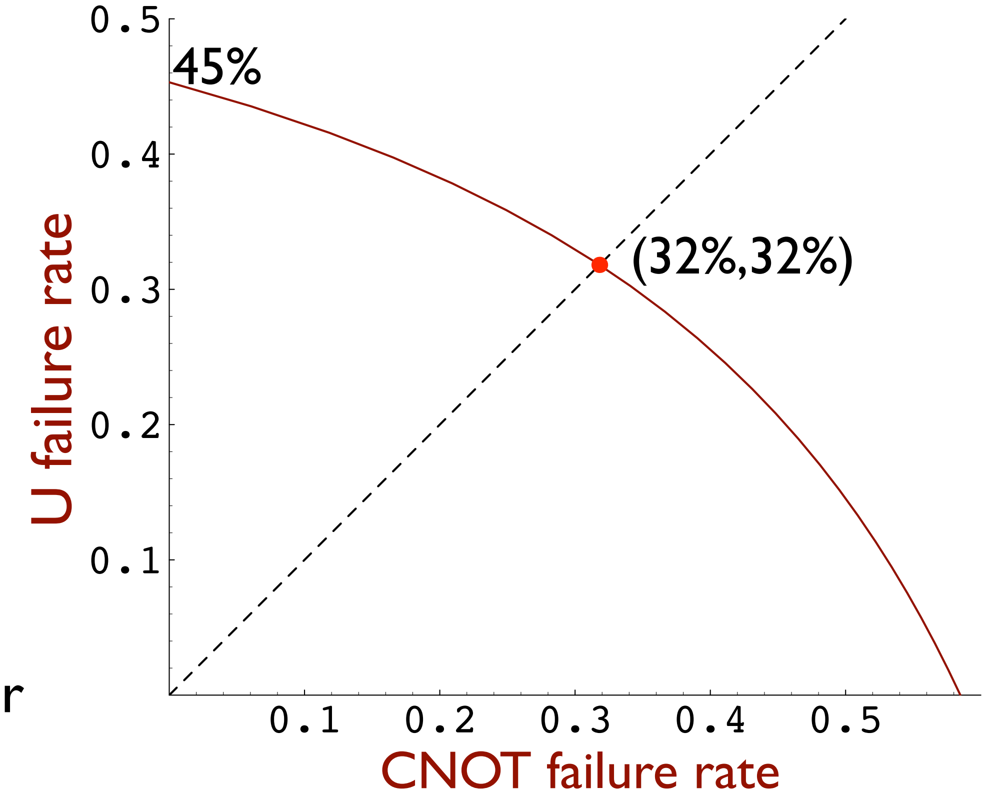
1. Compute polyhedron convex hull of two-qubit stabilizer states.
2. Compute unitary U which accepts most noise before $(I \otimes \mathcal{E})|\Psi\rangle$ is a mixture of stabilizers.

Answer: π/8 gate takes most, 45% depolarizing, so 45% upper-bounds FT threshold.
By **P/D** algorithm on $(I \otimes \mathcal{E})|\Psi\rangle$, this is tight.

Note: Sufficient to compute convex hull of stabilizer states arising from J. isom.

Application 2: FT threshold upper bounds

- BCLLSU '06:
  - Compute polyhedron convex hull of two-qubit stabilizer states
  - Compute one-qubit U which takes most noise before J. isom. gives mixture of stabilizers.
  - Answer: π/8 gate takes most, 45% depolarizing noise



- Open question 1: Better bounds assuming noisy stabilizer op's
  - 45% is tight with perfect stabilizer op's, but too conservative o.w.
  - First U applied must be to noisy stabilizer state. Using that state requires more noisy stabilizer op's. ∴ Get stabilizer mixture with less noise on U.
  - By how much can this improve FT threshold upper bound?
- Open question 2: Are there better upper bounds — i.e., do non-stabilizer states which are not distillable exist?

Better distillation upper bounds?

- Can we prove better upper bounds on distillability (and FT threshold), beyond the Gottesman-Knill limit?
- One possible approach: Reduce to single-qubit case.
  - Theorem: An n-qubit *pure* state $|\psi\rangle$ is distillable $\Longleftrightarrow$ one $|\psi\rangle$ copy can be reduced to a single-qubit distillable (pure) state. (Every n-qubit non-stabilizer pure state is distillable.)

Better distillation upper bounds?

- Can we prove better upper bounds on distillability (and FT threshold), beyond the Gottesman-Knill limit?
- One possible approach: Reduce to single-qubit case.
  - Theorem: An n-qubit *pure* state $|\psi\rangle$ is distillable $\Leftrightarrow$ one $|\psi\rangle$ copy can be reduced to a single-qubit distillable (pure) state.
  - Same holds for all previously proposed multi-qubit mixed ancilla states, either arising from the Jamiolkowski isomorphism [VHP '05, BCLLSU '06], or Dennis's $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle\right)$

    I.e., reductions to nonstabilizer single-qubit states exist for all noise values up to until the states become a mixture of stabilizer states
  - Could this hold generally?

An interesting two-qubit state

- One possible approach: Reduce to single-qubit case.
  - Theorem: An n-qubit *pure* state $|\psi\rangle$ is distillable $\Leftrightarrow$ one $|\psi\rangle$ copy can be reduced to a single-qubit distillable (pure) state.
  - Could this hold generally?
- **No**. There exist two-qubit states which are not mixtures of stabilizer states, but for which every 2-to-1-qubit stabilizer reduction outputs a stabilizer state mixture.

$$\frac{1}{4}\mathrm{II} + \frac{1}{12}\left(\mathrm{IY} + \mathrm{IZ} - \mathrm{XX} + \mathrm{YX} + \mathrm{ZX}\right)$$

- In fact, there are eight inequivalent faces of the polyhedron, for only one of them do 2-to-1-qubit stabilizer reductions exist.
- Among the seven other classes of examples, this has the most structure (e.g., nonzero Pauli coordinates all anticommute), making it perhaps the most promising for proving undistillable.

Conclusion

- Magic states distillation has tight connections to fault-tolerance.
  - Distillation upper bounds give FT upper bounds.
  - Distillation lower bounds help FT lower bounds.
- Open problems: Better bounds
  - Better *stable* distillation procedures for FT.
  - Better understanding of multi-qubit case. In particular, can

$$\frac{1}{4}\mathrm{II} + \frac{1}{12}\left(\mathrm{IY} + \mathrm{IZ} - \mathrm{XX} + \mathrm{YX} + \mathrm{ZX}\right)$$

    be distilled? What are the two-qubit "magic" states analogous to H and T?

- Details: quant-ph/0608085 and Ch. 6 of quant-ph/0612004

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