CSCI 420 Computer Graphics
Lecture 4

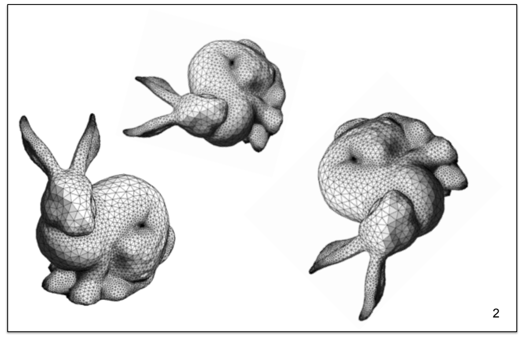
# Transformations

Vector Spaces
Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 4]

Jernej Barbic
University of Southern California

1

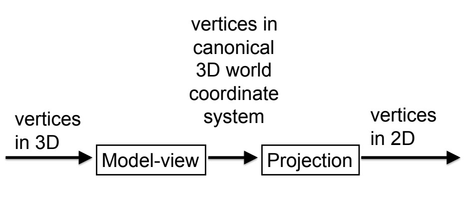
## OpenGL Transformations



2

## OpenGL Transformation Matrices

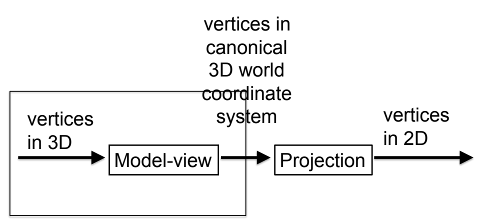
- Model-view matrix (4x4 matrix)
- Projection matrix (4x4 matrix)




3

## 4x4 Model-view Matrix (this lecture)

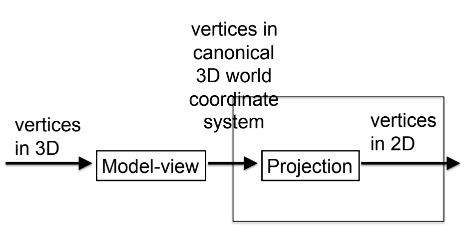
- Translate, rotate, scale objects
- Position the camera




4

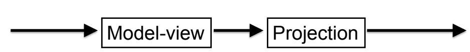
## 4x4 Projection Matrix (next lecture)

- Project from 3D to 2D




5

## OpenGL Transformation Matrices




- Manipulated separately in OpenGL (must set matrix mode) :

```
glMatrixMode (GL_MODELVIEW);
glMatrixMode (GL_PROJECTION);
```

6

## Setting the Current Model-view Matrix

- Load or post-multiply

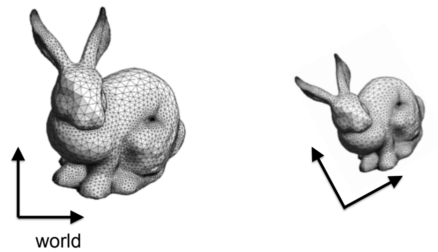
```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity(); // very common usage
float m[16] = { … };
glLoadMatrixf(m); // rare, advanced
glMultMatrixf(m); // rare, advanced
```

- Use library functions

```
glTranslatef(dx, dy, dz);
glRotatef(angle, vx, vy, vz);
glScalef(sx, sy, sz);
```

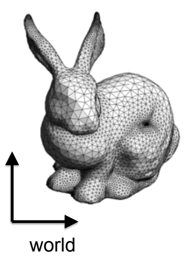
7

## Translated, rotated, scaled object




8

## The *rendering* coordinate system



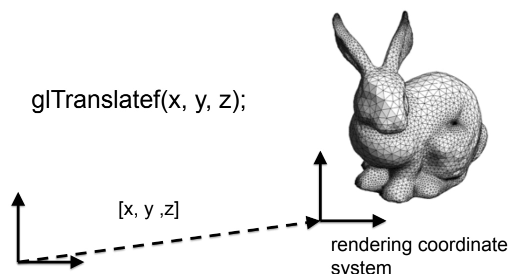

Initially (after glLoadIdentity()) :

rendering coordinate system = world coordinate system

9

## The *rendering* coordinate system

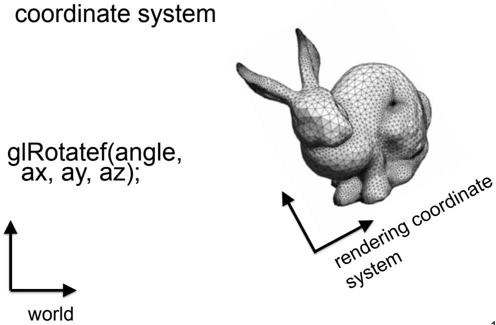
glTranslatef(x, y, z);




10

## The *rendering* coordinate system

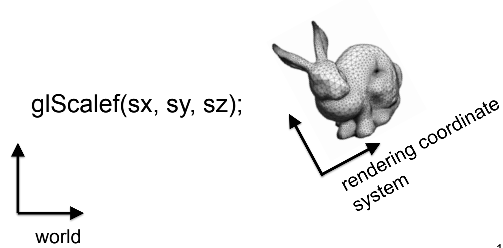
glRotatef(angle, ax, ay, az);




11

## The *rendering* coordinate system

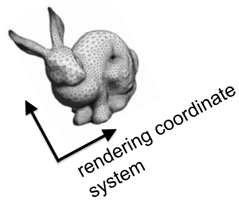
glScalef(sx, sy, sz);

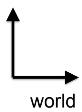



12

## OpenGL code

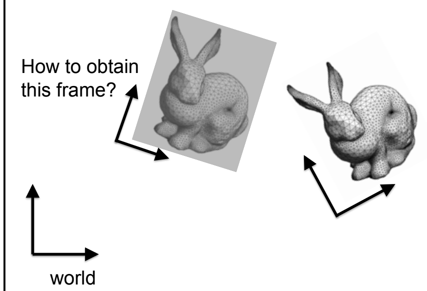
```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity();
glTranslatef(x, y, z);
glRotatef(angle, ax, ay, az);
glScalef(sx, sy, sz);
renderBunny();
```




13

## Rendering more objects

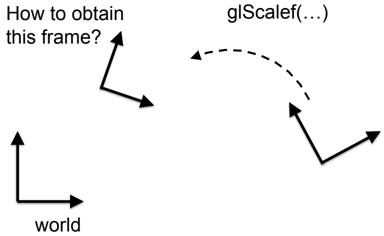



14

## Solution 1:

Find glTranslate(…), glRotatef(…), glScalef(…)

How to obtain this frame?



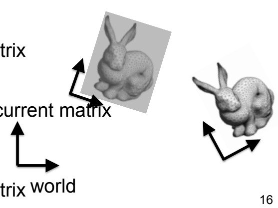

15

## Solution 2: gl{Push,Pop}Matrix

```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity();

// render first bunny
glPushMatrix(); // store current matrix
glTranslate3f(…);
glRotatef(…);
renderBunny();
glPopMatrix(); // pop matrix

// render second bunny
glPushMatrix(); // store current matrix
glTranslate3f(…);
glRotatef(…);
renderBunny();
glPopMatrix(); // pop matrix
```




16

# 3D Math Review

17

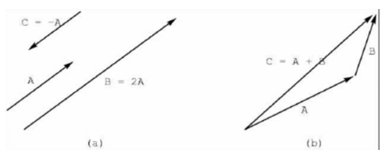
## Scalars

- Scalars $\alpha$, $\beta$, $\gamma$ from a *scalar field*
- Operations $\alpha+\beta$, $\alpha \cdot \beta$, 0, 1, $-\alpha$, $(\ )^{-1}$
- “Expected” laws apply
- Examples: rationals or reals with addition and multiplication

18

## Vectors

- Vectors *u, v, w* from a *vector space*
- Vector addition *u + v* , subtraction *u - v*
- Zero vector **0**
- Scalar multiplication $\alpha\, v$




19

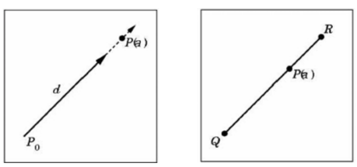
## Euclidean Space

- Vector space over real numbers
- Three-dimensional in computer graphics
- Dot product: $\alpha = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- *u, v* are *orthogonal* if $u \cdot v = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of *v*

20

## Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha\, d$



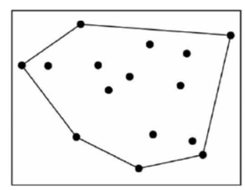

- Line segment between *Q* and *R*:
  $P(\alpha) = (1-\alpha)\, Q + \alpha\, R$ for $0 \le \alpha \le 1$

21

## Convex Hull

- Convex hull defined by

  $P = \alpha_1 P_1 + \ldots + \alpha_n P_n$
  for $\alpha_1 + \ldots + \alpha_n = 1$
  and $0 \le \alpha_i \le 1,\ i = 1, \ldots, n$

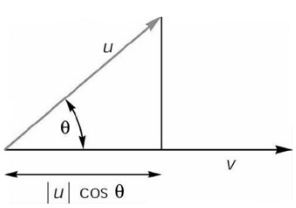


22

## Projection

- Dot product projects one vector onto another vector

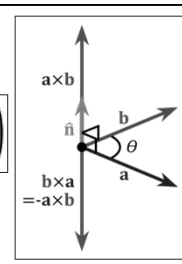
  $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\,|v| \cos(\theta)$
  $pr_v\, u = (u \cdot v)\, v / |v|^2$



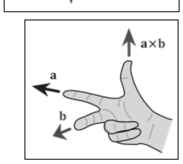

23

## Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$



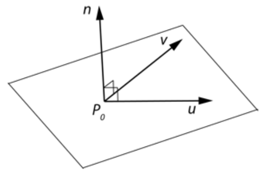

- $|a \times b| = |a|\,|b|\,|\sin(\theta)|$
- Cross product is perpendicular to both a and b
- Right-hand rule




Source: Wikipedia

## Plane

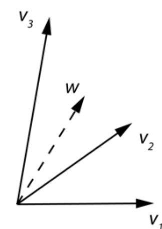
- Plane defined by point $P_0$ and vectors u and v
- u and v should not be parallel
- Parametric form:
  $T(\alpha, \beta) = P_0 + \alpha u + \beta v$
  ($\alpha$ and $\beta$ are scalars)
- $n = u \times v / |u \times v|$ is the normal
- $n \cdot (P - P_0) = 0$ if and only if P lies in plane




25

## Coordinate Systems

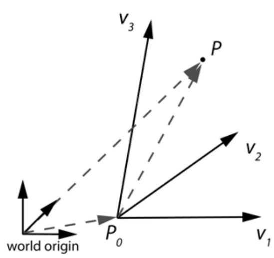
- Let $v_1$, $v_2$, $v_3$ be three linearly independent vectors in a 3-dimensional vector space
- Can write *any* vector w as
  $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
  for some scalars $\alpha_1, \alpha_2, \alpha_3$




26

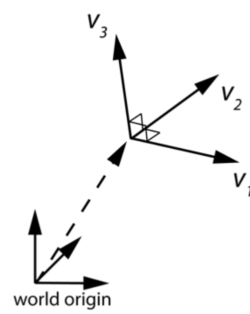
## Frames

- Frame = origin $P_0$ + coordinate system
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$




27

## In Practice, Frames are Often Orthogonal




28

## Change of Coordinate System

- Bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$
- Express basis vectors $u_i$ in terms of $v_j$

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$
$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$
$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

- Represent in matrix form:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

29

# Representing 3D transformations (and model-view matrices)

30

## Linear Transformations

- 3 x 3 matrices represent linear transformations
  **a = Mb**
- Can represent rotation, scaling, and reflection
- Cannot represent translation

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

31

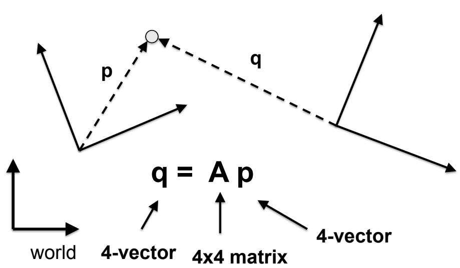
## In order to represent rotations, scales AND translations: Homogeneous Coordinates

- Augment $[\alpha_1 \;\; \alpha_2 \;\; \alpha_3]^T$ by adding a fourth component (1):
  **p** = $[\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T$

- Homogeneous property:
  **p** = $[\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T = [\beta\alpha_1 \;\; \beta\alpha_2 \;\; \beta\alpha_3 \;\; \beta]^T$ ,
  for any scalar $\beta \neq 0$

32

## Homogeneous coordinates are transformed by 4x4 matrices




33

## Affine Transformations (4x4 matrices)

- Translation
- Rotation
- Scaling
- Any composition of the above
- Later: projective (perspective) transformations
  - Also expressible as 4 x 4 matrices!

34

## Translation

- **q** = **p** + **d** where **d** = $[\alpha_x \;\; \alpha_y \;\; \alpha_z \;\; 0]^T$
- **p** = $[x \;\; y \;\; z \;\; 1]^T$
- **q** = $[x' \;\; y' \;\; z' \;\; 1]^T$
- Express in matrix form **q** = **T p** and solve for **T**

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

35

## Scaling

- $x' = \beta_x x$
- $y' = \beta_y y$
- $z' = \beta_z z$
- Express as **q** = **S p** and solve for **S**

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

36

## Rotation in 2 Dimensions

- Rotation by θ about the origin
- x' = x cos θ – y sin θ
- y' = x sin θ + y cos θ

- Express in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note that the determinant is 1

37

## Rotation in 3 Dimensions

- Orthogonal matrices:

$RR^T = R^TR = I$
det($R$) = 1

- Affine transformation:

$$A = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

38

## Affine Matrices are Composed by Matrix Multiplication

- $\mathbf{A = A_1 A_2 A_3}$
- Applied from right to left
- $\mathbf{A}\ \mathbf{p} = (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)\ \mathbf{p} = \mathbf{A}_1\ (\mathbf{A}_2\ (\mathbf{A}_3\ p))$
- When calling glTranslate3f, glRotatef, or glScalef, OpenGL forms the corresponding 4x4 matrix, and multiplies the current modelview matrix with it.

39

## Summary

- OpenGL Transformation Matrices
- Vector Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices

40