

## Geometric Queries for Ray Tracing

Ray-Surface Intersection  
Barycentric Coordinates  
[Ch. 13.2 - 13.3]

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## Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
  - Spheres
  - Planes
  - Polygons
  - Quadrics

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## Intersection of Rays and Parametric Surfaces

- Ray in parametric form
  - Origin  $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
  - Direction  $\mathbf{d} = [x_d \ y_d \ z_d]^T$
  - Assume  $\mathbf{d}$  is normalized ( $x_d^2 + y_d^2 + z_d^2 = 1$ )
  - Ray  $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$  for  $t > 0$
- Surface in parametric form
  - Point  $\mathbf{q} = \mathbf{g}(u, v)$ , possible bounds on  $u, v$
  - Solve  $\mathbf{p} + \mathbf{d} t = \mathbf{g}(u, v)$
  - Three equations in three unknowns ( $t, u, v$ )

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## Intersection of Rays and Implicit Surfaces

- Ray in parametric form
  - Origin  $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
  - Direction  $\mathbf{d} = [x_d \ y_d \ z_d]^T$
  - Assume  $\mathbf{d}$  normalized ( $x_d^2 + y_d^2 + z_d^2 = 1$ )
  - Ray  $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$  for  $t > 0$
- Implicit surface
  - Given by  $f(\mathbf{q}) = 0$
  - Consists of all points  $\mathbf{q}$  such that  $f(\mathbf{q}) = 0$
  - Substitute ray equation for  $\mathbf{q}$ :  $f(\mathbf{p}_0 + \mathbf{d} t) = 0$
  - Solve for  $t$  (univariate root finding)
  - Closed form (if possible), otherwise numerical approximation

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## Ray-Sphere Intersection I

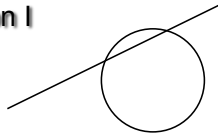
- Common and easy case
- Define sphere by
  - Center  $\mathbf{c} = [x_c \ y_c \ z_c]^T$
  - Radius  $r$
  - Surface  $f(\mathbf{q}) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
- Plug in ray equations for  $x, y, z$ :

$$x = x_0 + x_d t, \quad y = y_0 + y_d t, \quad z = z_0 + z_d t$$

- And we obtain a scalar equation for  $t$ :

$$(x_0 + x_d t - x_c)^2 + (y_0 + y_d t - y_c)^2 + (z_0 + z_d t - z_c)^2 = r^2$$

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## Ray-Sphere Intersection II

- Simplify to

$$at^2 + bt + c = 0$$

where

$$a = x_d^2 + y_d^2 + z_d^2 = 1 \quad \text{since } |d| = 1$$

$$b = 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c))$$

$$c = (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2$$

- Solve to obtain  $t_0$  and  $t_1$

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \quad \text{Check if } t_0, t_1 > 0 \text{ (ray)}$$

Return  $\min(t_0, t_1)$

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### Ray-Sphere Intersection III

- For lighting, calculate unit normal

$$n = \frac{1}{r} [(x_i - x_c) \quad (y_i - y_c) \quad (z_i - z_c)]^T$$

- Negate if ray originates inside the sphere!
- Note possible problems with roundoff errors

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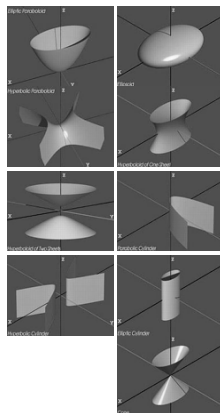
### Simple Optimizations

- Factor common subexpressions
- Compute only what is necessary
  - Calculate  $b^2 - 4c$ , abort if negative
  - Compute normal only for closest intersection
  - Other similar optimizations

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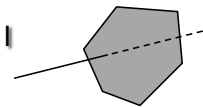
### Ray-Quadric Intersection

- Quadric  $f(\mathbf{p}) = f(x, y, z) = 0$ , where  $f$  is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG



### Ray-Polygon Intersection I

- Assume planar polygon in 3D
  1. Intersect ray with plane containing polygon
  2. Check if intersection point is inside polygon
- Plane
  - Implicit form:  $ax + by + cz + d = 0$
  - Unit normal:  $\mathbf{n} = [a \quad b \quad c]^T$  with  $a^2 + b^2 + c^2 = 1$



- Substitute:
 
$$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$
- Solve:
 
$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$

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### Ray-Polygon Intersection II

- Substitute  $t$  to obtain intersection point in plane
- Rewrite using dot product
 
$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(\mathbf{n} \cdot \mathbf{p}_0 + d)}{\mathbf{n} \cdot \mathbf{d}}$$
- If  $\mathbf{n} \cdot \mathbf{d} = 0$ , no intersection (ray parallel to plane)
- If  $t \leq 0$ , the intersection is behind ray origin

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### Test if point inside polygon

- Use even-odd rule or winding rule
- Easier if polygon is in 2D (project from 3D to 2D)
- Easier for triangles (tessellate polygons)

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### Point-in-triangle testing

- Critical for polygonal models
- Project the triangle, and point of plane intersection, onto one of the planes  $x = 0$ ,  $y = 0$ , or  $z = 0$  (pick a plane not perpendicular to triangle) (such a choice always exists)
- Then, do the 2D test in the plane, by computing barycentric coordinates (follows next)

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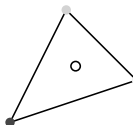
### Outline

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates

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### Interpolated Shading for Ray Tracing

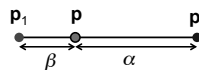
- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- Barycentric coordinates
- Yields same answer as scan conversion



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### Barycentric Coordinates in 1D

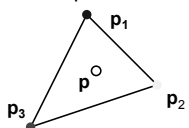
- Linear interpolation
  - $p(t) = (1 - t)p_1 + t p_2, 0 \leq t \leq 1$
  - $p(t) = \alpha p_1 + \beta p_2$  where  $\alpha + \beta = 1$
  - $p$  is between  $p_1$  and  $p_2$  iff  $0 \leq \alpha, \beta \leq 1$
- Geometric intuition
  - Weigh each vertex by ratio of distances from ends



- $\alpha, \beta$  are called barycentric coordinates

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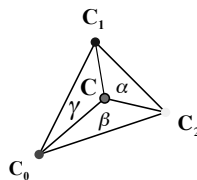
### Barycentric Coordinates in 2D

- Now, we have 3 points instead of 2
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- Define 3 barycentric coordinates,  $\alpha, \beta, \gamma$
  - $p = \alpha p_1 + \beta p_2 + \gamma p_3$
  - $p$  inside triangle iff  $0 \leq \alpha, \beta, \gamma \leq 1, \alpha + \beta + \gamma = 1$
  - How do we calculate  $\alpha, \beta, \gamma$  given  $p$ ?

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### Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas



$$\alpha = \frac{\text{Area}(CC_1C_2)}{\text{Area}(C_0C_1C_2)}$$

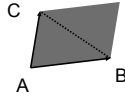
$$\beta = \frac{\text{Area}(C_0CC_2)}{\text{Area}(C_0C_1C_2)}$$

$$\gamma = \frac{\text{Area}(C_0C_1C)}{\text{Area}(C_0C_1C_2)} = 1 - \alpha - \beta$$

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.

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### Computing Triangle Area in 3D



- Use cross product
- Parallelogram formula
- $\text{Area}(ABC) = (1/2) |(B - A) \times (C - A)|$
- How to get correct sign for barycentric coordinates?
  - tricky, but possible:
    - compare directions of vectors  $(B - A) \times (C - A)$ , for triangles  $CC_1C_2$  vs  $C_0C_1C_2$ , etc. (either 0 (sign+) or 180 deg (sign-) angle)
    - easier alternative: project to 2D, use 2D formula
    - projection to 2D preserves barycentric coordinates

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### Computing Triangle Area in 2D

- Suppose we project the triangle to xy plane
- $\text{Area}(xy\text{-projection}(ABC)) = (1/2) ((b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y))$
- This formula gives correct sign (important for barycentric coordinates)

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### Summary

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates

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