

## Quaternions and Rotations

Rotations  
Quaternions  
Motion Capture  
[Angel Ch. 4.12]

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## Rotations

- Very important in computer animation and robotics
- Joint angles, rigid body orientations, camera parameters
- 2D or 3D

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## Rotations in Three Dimensions

- Orthogonal matrices:

$$RR^T = R^T R = I$$
$$\det(R) = 1$$

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

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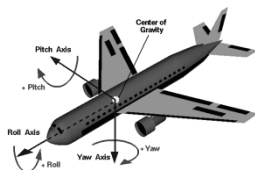
## Representing Rotations in 3D

- Rotations in 3D have essentially three parameters
- Axis + angle (2 DOFs + 1DOFs)
  - How to represent the axis?  
Longitude / latitude have singularities
- 3x3 matrix
  - 9 entries (redundant)

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## Representing Rotations in 3D

- Euler angles
  - roll, pitch, yaw
  - no redundancy (good)
  - gimbal lock singularities



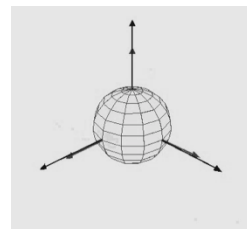
Source: Wikipedia

- Quaternions
  - generally considered the “best” representation
  - redundant (4 values), but only by one DOF (not severe)
  - stable interpolations of rotations possible

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## Euler Angles

1. Yaw  
rotate around y-axis
2. Pitch  
rotate around (rotated) x-axis
3. Roll  
rotate around (rotated) y-axis

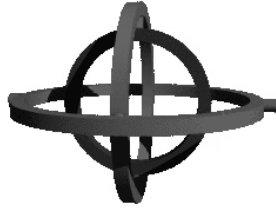


Source: Wikipedia

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## Gimbal Lock

When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.

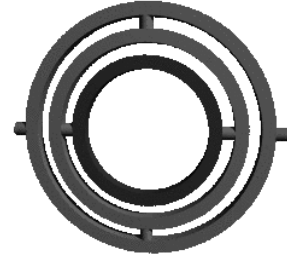


Source: Wikipedia

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Source: Wikipedia

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## Outline

- Rotations
- Quaternions
- Motion Capture

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## Quaternions

- Generalization of complex numbers

- Three imaginary numbers:  $i, j, k$

$$\begin{aligned} i^2 = -1, j^2 = -1, k^2 = -1, \\ ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j \end{aligned}$$

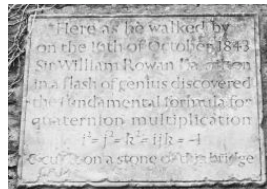
- $q = s + x i + y j + z k$ ,  $s, x, y, z$  are scalars

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## Quaternions

- Invented by Hamilton in 1843 in Dublin, Ireland

- Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication



Source: Wikipedia

$$i^2 = j^2 = k^2 = i j k = -1$$

& cut it on a stone of this bridge.

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## Quaternions

- Quaternions are **not** commutative!

$$q_1 q_2 \neq q_2 q_1$$

- However, the following hold:

$$\begin{aligned} (q_1 q_2) q_3 &= q_1 (q_2 q_3) \\ (q_1 + q_2) q_3 &= q_1 q_3 + q_2 q_3 \\ q_1 (q_2 + q_3) &= q_1 q_2 + q_1 q_3 \\ \alpha (q_1 + q_2) &= \alpha q_1 + \alpha q_2 \quad (\alpha \text{ is scalar}) \\ (\alpha q_1) q_2 &= \alpha (q_1 q_2) = q_1 (\alpha q_2) \quad (\alpha \text{ is scalar}) \end{aligned}$$

- I.e. all usual manipulations are valid, except cannot reverse multiplication order.

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## Quaternions

- Exercise: multiply two quaternions

$$(2 - i + j + 3k)(-1 + i + 4j - 2k) = \dots$$

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## Quaternion Properties

- $q = s + x i + y j + z k$
- Norm:  $|q|^2 = s^2 + x^2 + y^2 + z^2$
- Conjugate quaternion:  $\bar{q} = s - x i - y j - z k$
- Inverse quaternion:  $q^{-1} = \bar{q} / |q|^2$
- Unit quaternion:  $|q| = 1$
- Inverse of unit quaternion:  $q^{-1} = \bar{q}$

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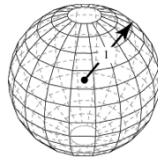
## Quaternions and Rotations

- Rotations are represented by **unit** quaternions

$$q = s + x i + y j + z k$$

$$s^2 + x^2 + y^2 + z^2 = 1$$

- Unit quaternion sphere (unit sphere in 4D)



Source:  
Wolfram Research  
unit sphere  
in 4D

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## Rotations to Unit Quaternions

- Let (unit) rotation axis be  $[u_x, u_y, u_z]$ , and angle  $\theta$
- Corresponding quaternion is
 
$$q = \cos(\theta/2) + \sin(\theta/2) u_x i + \sin(\theta/2) u_y j + \sin(\theta/2) u_z k$$
- Composition of rotations  $q_1$  and  $q_2$  equals  $q = q_2 q_1$
- 3D rotations do not commute!

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## Unit Quaternions to Rotations

- Let  $v$  be a (3-dim) vector and let  $q$  be a unit quaternion
- Then, the corresponding rotation transforms vector  $v$  to  $q v q^{-1}$   
( $v$  is a quaternion with scalar part equaling 0, and vector part equaling  $v$ )

$$\text{For } q = a + b i + c j + d k$$

$$R = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

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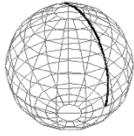
## Quaternions

- Quaternions  $q$  and  $-q$  give the same rotation!
- Other than this, the relationship between rotations and quaternions is unique

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## Quaternion Interpolation

- Better results than Euler angles
- A quaternion is a point on the 4-D unit sphere
  - interpolating rotations requires a unit quaternion at each step -- another point on the 4-D sphere
  - move with constant angular velocity along the great circle between the two points
  - Spherical Linear intERPolation (SLERPing)
- Any rotation is given by 2 quaternions, so pick the shortest SLERP



Source:  
Wolfram Research

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## Quaternion Interpolation

- To interpolate more than two points:
  - solve a non-linear variational constrained optimization (numerically)
- Further information: Ken Shoemake in the SIGGRAPH '85 proceedings (Computer Graphics, V. 19, No. 3, P.245)

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## What is Motion Capture?

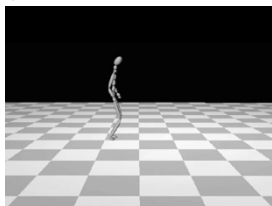
- Motion capture is the process of tracking real-life motion in 3D and recording it for use in any number of applications.



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## Why Motion Capture?

- Keyframes are generated by instruments measuring a human performer — they do not need to be set manually
- The details of human motion such as style, mood, and shifts of weight are reproduced with little effort



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## Mocap Technologies: Optical

- Multiple high-resolution, high-speed cameras
- Light bounced from camera off of reflective markers
- High quality data
- Markers placeable anywhere
- Lots of work to extract joint angles
- Occlusion
- Which marker is which? (correspondence problem)
- 120-240 Hz @ 1Megapixel



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## Facial Motion Capture



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## Mocap Technologies: Electromagnetic

- Sensors give both position and orientation
- No occlusion or correspondence problem
- Little post-processing
- Limited accuracy



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## Mocap Technologies: Exoskeleton

- Really Fast (~500Hz)
- No occlusion or correspondence problem
- Little error
- Movement restricted
- Fixed sensors



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## Motion Capture

- Why not?
  - Difficult for non-human characters
    - Can you move like a hamster / duck / eagle ?
    - Can you capture a hamster's motion?
  - Actors needed
    - Which is more economical:
      - Paying an animator to place keys
      - Hiring a Martial Arts Expert

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## When to use Motion Capture?

- Complicated character motion
  - Where "uncomplicated" ends and "complicated" begins is up to question
  - A walk cycle is often more easily done by hand
  - A Flying Monkey Kick might be worth the overhead of mocap
- Can an actor better express character personality than the animator?

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## Summary

- Rotations
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