

CSCI 420 Computer Graphics
Lecture 20

Quaternions and Rotations

Rotations
Quaternions
Motion Capture
[Angel Ch. 4.12]

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Rotations

- Very important in computer animation and robotics
- Joint angles, rigid body orientations, camera parameters
- 2D or 3D

Rotations in Three Dimensions

- Orthogonal matrices:

$$RR^T = R^T R = I$$
$$\det(R) = 1$$

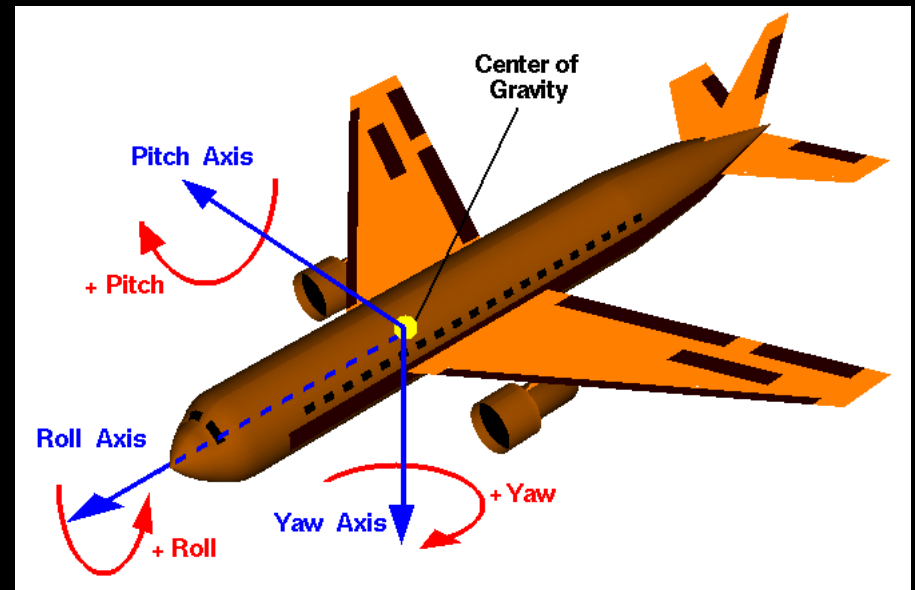
$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Representing Rotations in 3D

- Rotations in 3D have essentially three parameters
- Axis + angle (2 DOFs + 1DOFs)
 - How to represent the axis?
Longitude / latitude have singularities
- 3x3 matrix
 - 9 entries (redundant)

Representing Rotations in 3D

- Euler angles
 - roll, pitch, yaw
 - no redundancy (good)
 - gimbal lock singularities

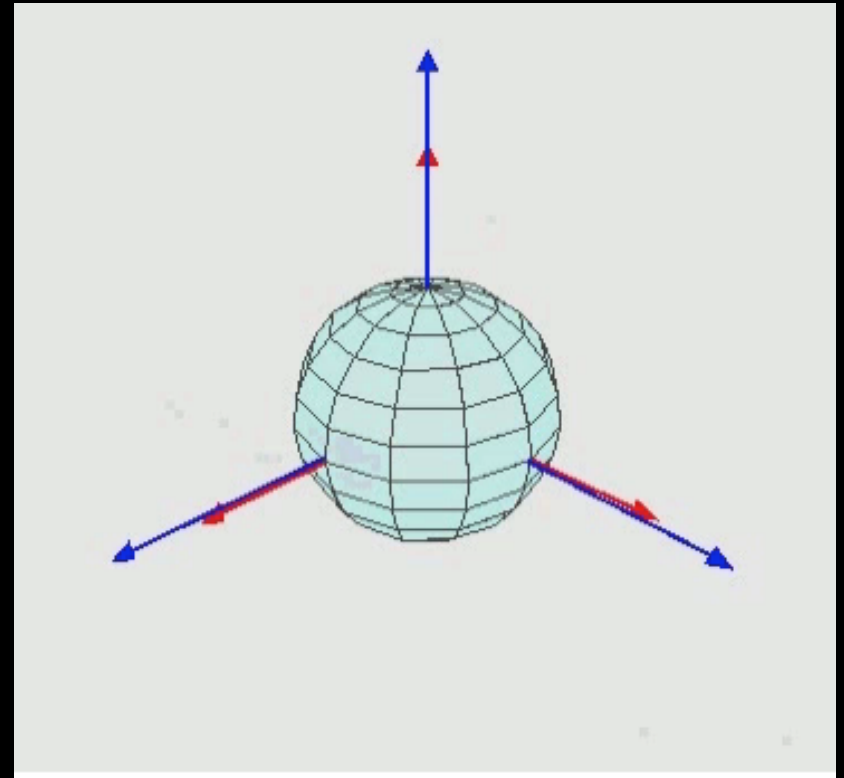


Source: Wikipedia

- Quaternions
 - generally considered the “best” representation
 - redundant (4 values), but only by one DOF (not severe)
 - stable interpolations of rotations possible

Euler Angles

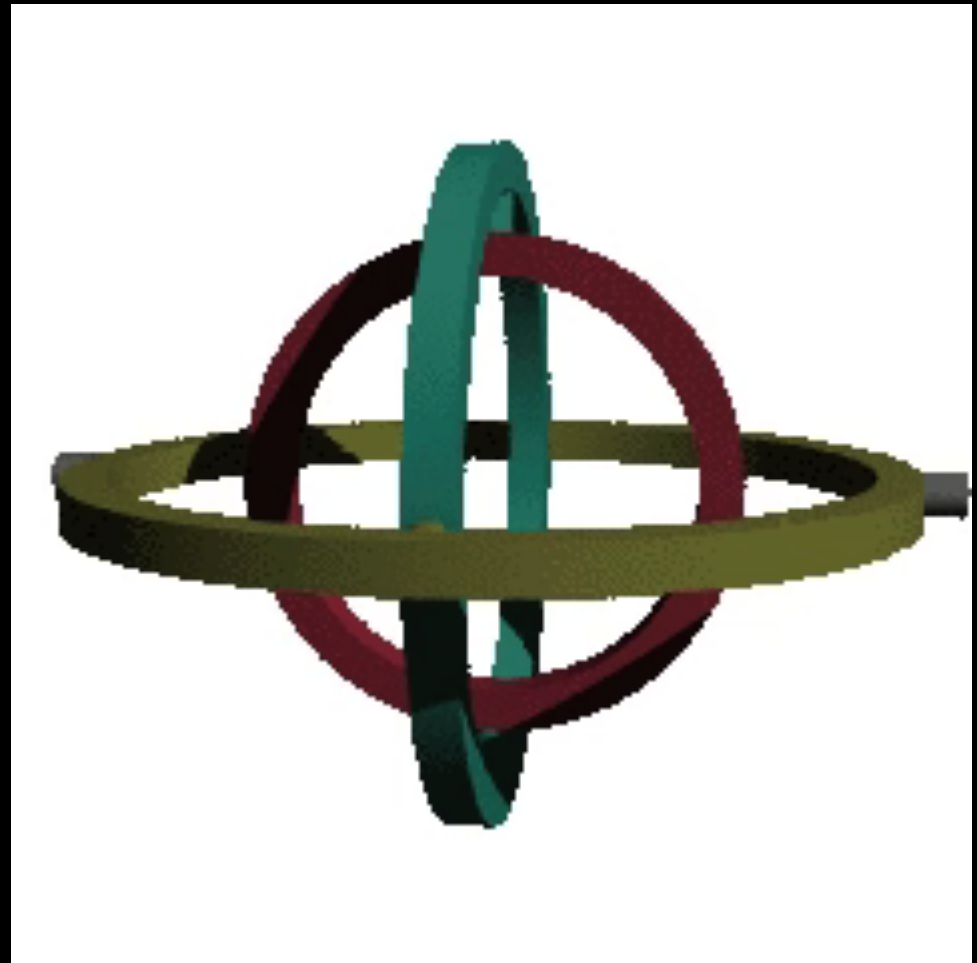
1. **Yaw**
rotate around y-axis
2. **Pitch**
rotate around (rotated) x-axis
3. **Roll**
rotate around (rotated) y-axis



Source: Wikipedia

Gimbal Lock

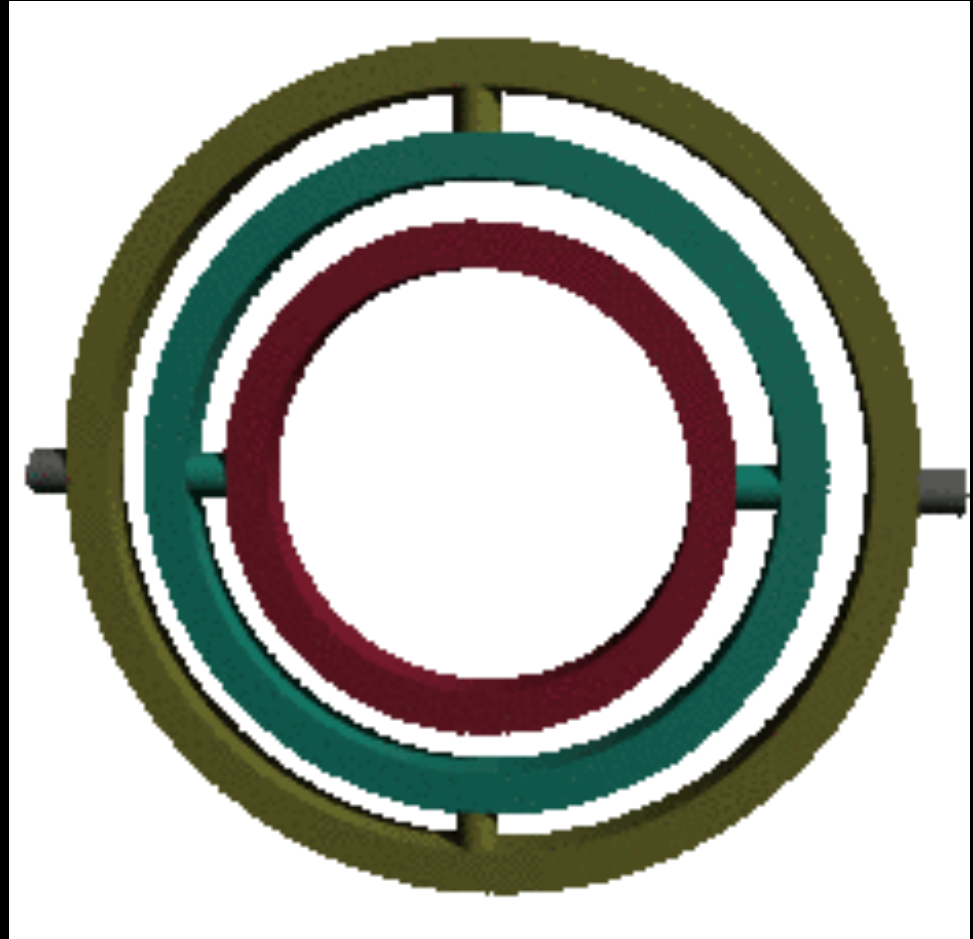
When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.



Source: Wikipedia

Gimbal Lock

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Source: Wikipedia

Outline

- Rotations
- **Quaternions**
- Motion Capture

Quaternions

- Generalization of complex numbers
- Three imaginary numbers: i, j, k

$$i^2 = -1, j^2 = -1, k^2 = -1,$$

$$ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$$

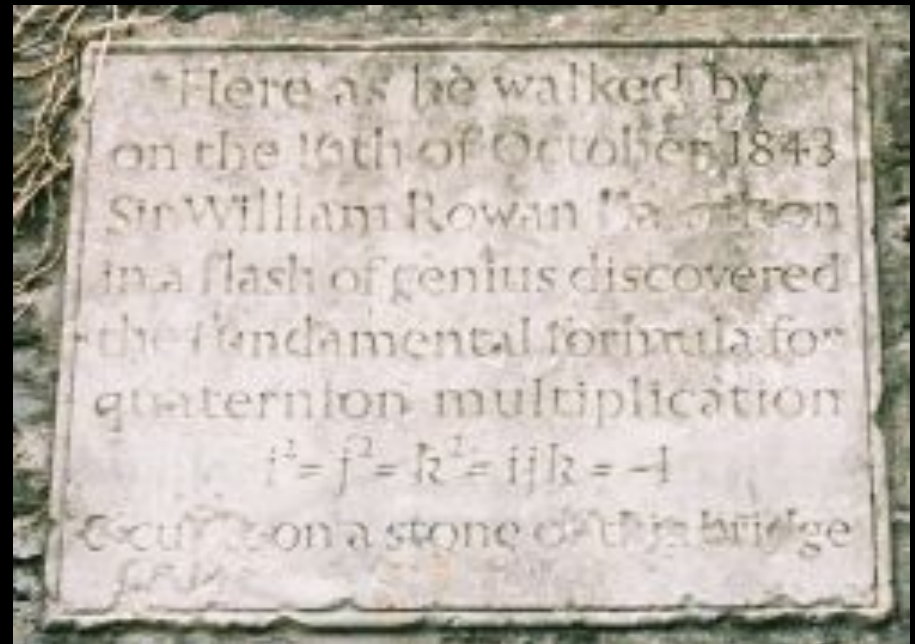
- $q = s + x i + y j + z k,$ s, x, y, z are scalars

Quaternions

- Invented by Hamilton in 1843 in Dublin, Ireland
- Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$i^2 = j^2 = k^2 = i j k = -1$$

& cut it on a stone of this bridge.



Source: Wikipedia

Quaternions

- Quaternions are **not** commutative!

$$q_1 q_2 \neq q_2 q_1$$

- However, the following hold:

$$(q_1 q_2) q_3 = q_1 (q_2 q_3)$$

$$(q_1 + q_2) q_3 = q_1 q_3 + q_2 q_3$$

$$q_1 (q_2 + q_3) = q_1 q_2 + q_1 q_3$$

$$\alpha (q_1 + q_2) = \alpha q_1 + \alpha q_2 \quad (\alpha \text{ is scalar})$$

$$(\alpha q_1) q_2 = \alpha (q_1 q_2) = q_1 (\alpha q_2) \quad (\alpha \text{ is scalar})$$

- I.e. all usual manipulations are valid, except cannot reverse multiplication order.

Quaternions

- Exercise: multiply two quaternions

$$(2 - i + j + 3k) (-1 + i + 4j - 2k) = \dots$$

Quaternion Properties

- $q = s + x i + y j + z k$
- Norm: $|q|^2 = s^2 + x^2 + y^2 + z^2$
- Conjugate quaternion: $\bar{q} = s - x i - y j - z k$
- Inverse quaternion: $q^{-1} = \bar{q} / |q|^2$
- Unit quaternion: $|q| = 1$
- Inverse of unit quaternion: $q^{-1} = \bar{q}$

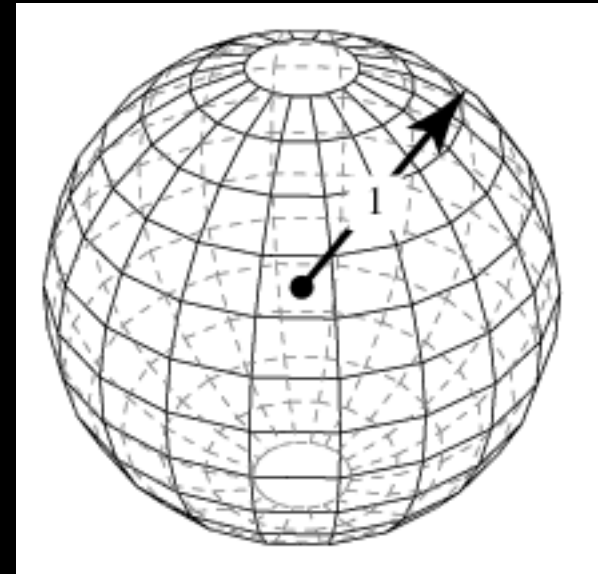
Quaternions and Rotations

- Rotations are represented by *unit* quaternions

- $q = s + x i + y j + z k$

$$s^2 + x^2 + y^2 + z^2 = 1$$

- Unit quaternion sphere
(unit sphere in 4D)



Source:
Wolfram Research

unit sphere
in 4D

Rotations to Unit Quaternions

- Let (unit) rotation axis be $[u_x, u_y, u_z]$, and angle θ
- Corresponding quaternion is

$$q = \cos(\theta/2) + \sin(\theta/2) u_x \mathbf{i} + \sin(\theta/2) u_y \mathbf{j} + \sin(\theta/2) u_z \mathbf{k}$$

- Composition of rotations q_1 and q_2 equals $q = q_2 q_1$
- 3D rotations do not commute!

Unit Quaternions to Rotations

- Let \mathbf{v} be a (3-dim) vector and let q be a unit quaternion
- Then, the corresponding rotation transforms vector \mathbf{v} to $q \mathbf{v} q^{-1}$

(\mathbf{v} is a quaternion with scalar part equaling 0, and vector part equaling \mathbf{v})

For $q = a + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}$

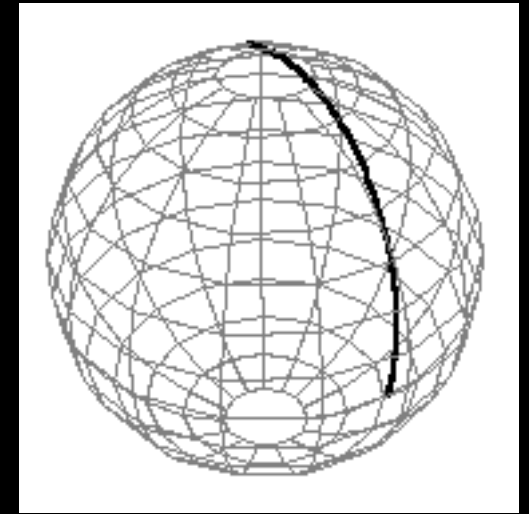
$$R = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & a^2 - b^2 + c^2 - d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & a^2 - b^2 - c^2 + d^2 \end{pmatrix}$$

Quaternions

- Quaternions q and $-q$ give the same rotation!
- Other than this, the relationship between rotations and quaternions is unique

Quaternion Interpolation

- Better results than Euler angles
- A quaternion is a point on the 4-D unit sphere
 - interpolating rotations requires a unit quaternion at each step -- another point on the 4-D sphere
 - move with constant angular velocity along the great circle between the two points
 - Spherical Linear intERPolation (**SLERP**ing)
- Any rotation is given by 2 quaternions, so pick the shortest SLERP



Source:
Wolfram Research

Quaternion Interpolation

- To interpolate more than two points:
 - solve a non-linear variational constrained optimization (numerically)
- Further information: Ken Shoemake in the SIGGRAPH '85 proceedings (Computer Graphics, V. 19, No. 3, P.245)

Outline

- Rotations
- Quaternions
- **Motion Capture**

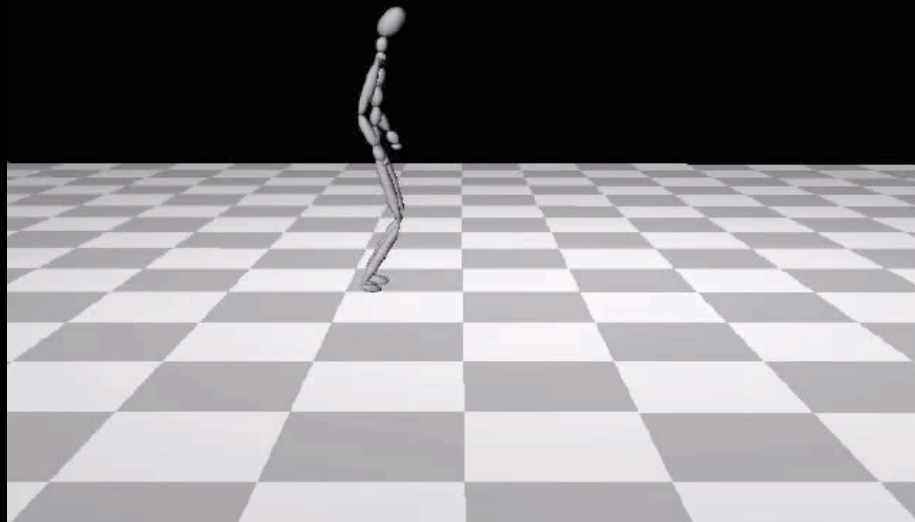
What is Motion Capture?

- Motion capture is the process of tracking real-life motion in 3D and recording it for use in any number of applications.



Why Motion Capture?

- Keyframes are generated by instruments measuring a human performer — they do not need to be set manually
- The details of human motion such as style, mood, and shifts of weight are reproduced with little effort



Mocap Technologies: Optical

- Multiple high-resolution, high-speed cameras
- Light bounced from camera off of reflective markers
- High quality data
- Markers placeable anywhere
- Lots of work to extract joint angles
- Occlusion
- Which marker is which?
(correspondence problem)
- 120-240 Hz @ 1Megapixel



Facial Motion Capture



Mocap Technologies: Electromagnetic

- Sensors give both position and orientation
- No occlusion or correspondence problem
- Little post-processing
- Limited accuracy



Mocap Technologies: Exoskeleton

- Really Fast (~500Hz)
- No occlusion or correspondence problem
- Little error
- Movement restricted
- Fixed sensors



Motion Capture

- Why not?
 - Difficult for non-human characters
 - Can you move like a hamster / duck / eagle ?
 - Can you capture a hamster's motion?
 - Actors needed
 - Which is more economical:
 - Paying an animator to place keys
 - Hiring a Martial Arts Expert

When to use Motion Capture?

- Complicated character motion
 - Where “uncomplicated” ends and “complicated” begins is up to question
 - A walk cycle is often more easily done by hand
 - A Flying Monkey Kick might be worth the overhead of mocap
- Can an actor better express character personality than the animator?

Summary

- Rotations
- Quaternions
- Motion Capture