

CSCI 420 Computer Graphics

Lecture 16

Geometric Queries for Ray Tracing

Ray-Surface Intersection

Barycentric Coordinates

[Angel Ch. 11]

Jernej Barbic

University of Southern California

Ray-Surface Intersections

- Necessary in ray tracing
- General implicit surfaces
- General parametric surfaces
- Specialized analysis for special surfaces
 - Spheres
 - Planes
 - Polygons
 - Quadrics

Intersection of Rays and Parametric Surfaces

- Ray in parametric form
 - Origin $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
 - Direction $\mathbf{d} = [x_d \ y_d \ z_d]^T$
 - Assume \mathbf{d} is normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
 - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$ for $t > 0$
- Surface in parametric form
 - Point $\mathbf{q} = g(u, v)$, possible bounds on u, v
 - Solve $\mathbf{p}_0 + \mathbf{d} t = g(u, v)$
 - Three equations in three unknowns (t, u, v)

Intersection of Rays and Implicit Surfaces

- Ray in parametric form
 - Origin $\mathbf{p}_0 = [x_0 \ y_0 \ z_0]^T$
 - Direction $\mathbf{d} = [x_d \ y_d \ z_d]^T$
 - Assume \mathbf{d} normalized ($x_d^2 + y_d^2 + z_d^2 = 1$)
 - Ray $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{d} t$ for $t > 0$
- Implicit surface
 - Given by $f(\mathbf{q}) = 0$
 - Consists of all points \mathbf{q} such that $f(\mathbf{q}) = 0$
 - Substitute ray equation for \mathbf{q} : $f(\mathbf{p}_0 + \mathbf{d} t) = 0$
 - Solve for t (univariate root finding)
 - Closed form (if possible), otherwise numerical approximation

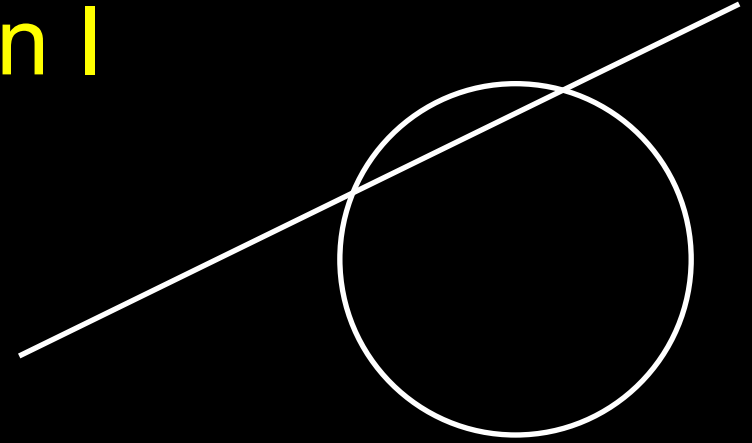
Ray-Sphere Intersection I

- Common and easy case
- Define sphere by
 - Center $\mathbf{c} = [x_c \ y_c \ z_c]^T$
 - Radius r
 - Surface $f(\mathbf{q}) = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - r^2 = 0$
- Plug in ray equations for x, y, z :

$$x = x_0 + x_{dt}, \quad y = y_0 + y_{dt}, \quad z = z_0 + z_{dt}$$

- And we obtain a scalar equation for t :

$$(x_0 + x_{dt} - x_c)^2 + (y_0 + y_{dt} - y_c)^2 + (z_0 + z_{dt} - z_c)^2 = r^2$$



Ray-Sphere Intersection II

- Simplify to

$$at^2 + bt + c = 0$$

where

$$\begin{aligned} a &= x_d^2 + y_d^2 + z_d^2 = 1 && \text{since } |d| = 1 \\ b &= 2(x_d(x_0 - x_c) + y_d(y_0 - y_c) + z_d(z_0 - z_c)) \\ c &= (x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2 - r^2 \end{aligned}$$

- Solve to obtain t_0 and t_1

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Check if $t_0, t_1 > 0$ (ray)
Return $\min(t_0, t_1)$

Ray-Sphere Intersection III

- For lighting, calculate unit normal

$$n = \frac{1}{r} [(x_i - x_c) \quad (y_i - y_c) \quad (z_i - z_c)]^T$$

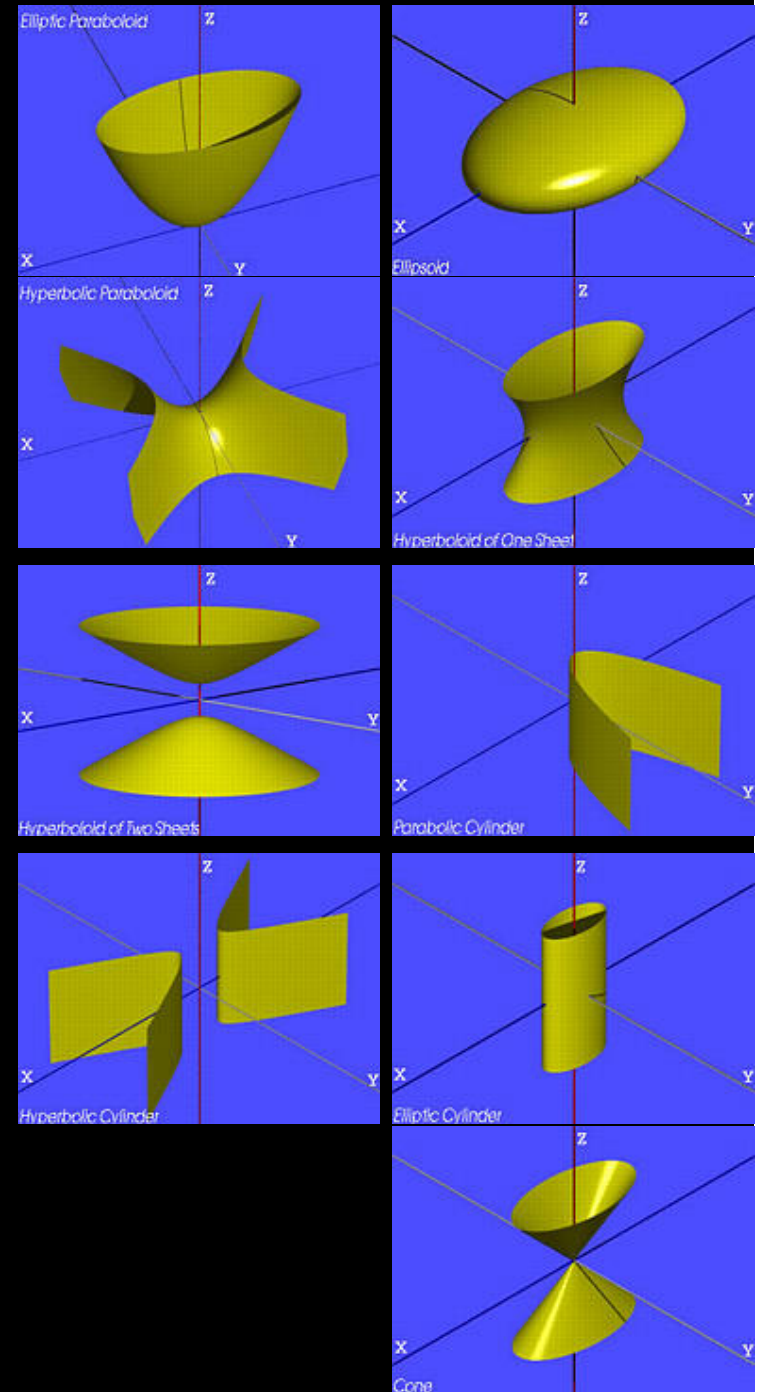
- Negate if ray originates inside the sphere!
- Note possible problems with roundoff errors

Simple Optimizations

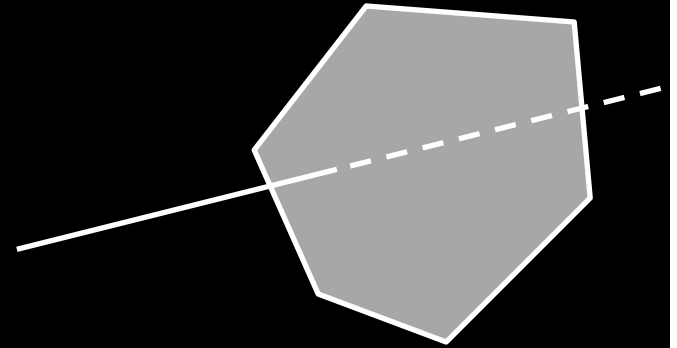
- Factor common subexpressions
- Compute only what is necessary
 - Calculate $b^2 - 4c$, abort if negative
 - Compute normal only for closest intersection
 - Other similar optimizations

Ray-Quadric Intersection

- Quadric $f(\mathbf{p}) = f(x, y, z) = 0$, where f is polynomial of order 2
- Sphere, ellipsoid, paraboloid, hyperboloid, cone, cylinder
- Closed form solution as for sphere
- Important case for modelling in ray tracing
- Combine with CSG



Ray-Polygon Intersection I



- Assume planar polygon in 3D
 1. Intersect ray with plane containing polygon
 2. Check if intersection point is inside polygon
- Plane
 - Implicit form: $ax + by + cz + d = 0$
 - Unit normal: $\mathbf{n} = [a \ b \ c]^T$ with $a^2 + b^2 + c^2 = 1$
- Substitute:

$$a(x_0 + x_d t) + b(y_0 + y_d t) + c(z_0 + z_d t) + d = 0$$

- Solve:

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d}$$

Ray-Polygon Intersection II

- Substitute t to obtain intersection point in plane
- Rewrite using dot product

$$t = \frac{-(ax_0 + by_0 + cz_0 + d)}{ax_d + by_d + cz_d} = \frac{-(n \cdot p_0 + d)}{n \cdot d}$$

- If $n \cdot d = 0$, no intersection (ray parallel to plane)
- If $t \leq 0$, the intersection is behind ray origin

Test if point inside polygon

- Use even-odd rule or winding rule
- Easier if polygon is in 2D
(project from 3D to 2D)
- Easier for triangles (tessellate polygons)

Point-in-triangle testing

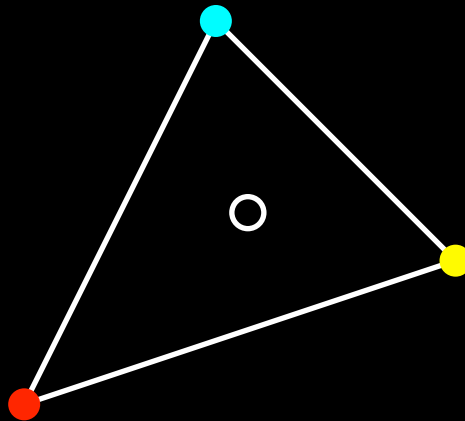
- Critical for polygonal models
- Project the triangle, and point of plane intersection, onto one of the planes $x = 0$, $y = 0$, or $z = 0$
(pick a plane not perpendicular to triangle)
(such a choice always exists)
- Then, do the 2D test in the plane, by computing barycentric coordinates
(follows next)

Outline

- Ray-Surface Intersections
- Special cases: sphere, polygon
- **Barycentric Coordinates**

Interpolated Shading for Ray Tracing

- Assume we know normals at vertices
- How do we compute normal of interior point?
- Need linear interpolation between 3 points
- **Barycentric coordinates**
- Yields same answer as scan conversion



Barycentric Coordinates in 1D

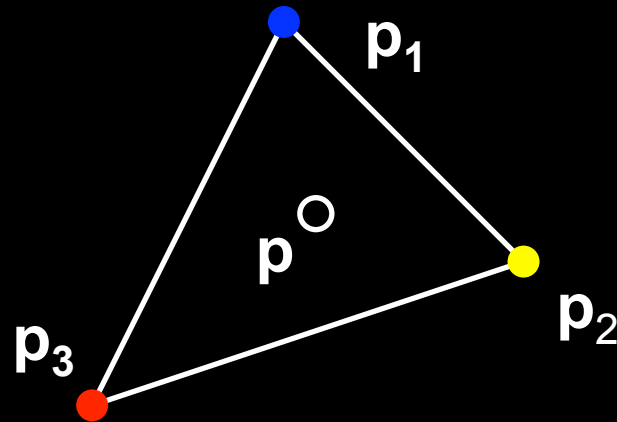
- Linear interpolation
 - $\mathbf{p}(t) = (1 - t)\mathbf{p}_1 + t \mathbf{p}_2, 0 \leq t \leq 1$
 - $\mathbf{p}(t) = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2$ where $\alpha + \beta = 1$
 - \mathbf{p} is between \mathbf{p}_1 and \mathbf{p}_2 iff $0 \leq \alpha, \beta \leq 1$
- Geometric intuition
 - Weigh each vertex by ratio of distances from ends



- α, β are called **barycentric coordinates**

Barycentric Coordinates in 2D

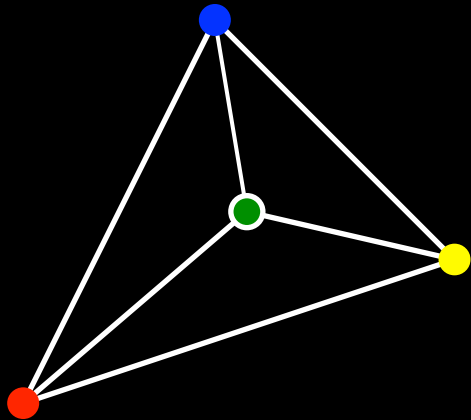
- Now, we have 3 points instead of 2



- Define 3 barycentric coordinates, α , β , γ
- $\mathbf{p} = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2 + \gamma \mathbf{p}_3$
- \mathbf{p} inside triangle iff $0 \leq \alpha, \beta, \gamma \leq 1, \alpha + \beta + \gamma = 1$
- How do we calculate α, β, γ given \mathbf{p} ?

Barycentric Coordinates for Triangle

- Coordinates are ratios of triangle areas



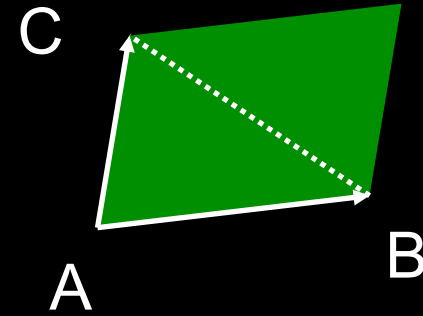
$$\alpha = \frac{\text{Area}(\mathbf{C}\mathbf{C}_1\mathbf{C}_2)}{\text{Area}(\mathbf{C}_0\mathbf{C}_1\mathbf{C}_2)}$$

$$\beta = \frac{\text{Area}(\mathbf{C}_0\mathbf{C}\mathbf{C}_2)}{\text{Area}(\mathbf{C}_0\mathbf{C}_1\mathbf{C}_2)}$$

$$\gamma = \frac{\text{Area}(\mathbf{C}_0\mathbf{C}_1\mathbf{C})}{\text{Area}(\mathbf{C}_0\mathbf{C}_1\mathbf{C}_2)} = 1 - \alpha - \beta$$

- Areas in these formulas should be signed, depending on clockwise (-) or anti-clockwise orientation (+) of the triangle! Very important for point-in-triangle test.

Computing Triangle Area in 3D



- Use cross product
- Parallelogram formula
- $\text{Area}(ABC) = (1/2) |(B - A) \times (C - A)|$
- How to get correct sign for barycentric coordinates?
 - tricky, but possible:
compare directions of vectors $(B - A) \times (C - A)$, for triangles CC_1C_2 vs $C_0C_1C_2$, etc.
(either 0 (sign+) or 180 deg (sign-) angle)
 - easier alternative: project to 2D, use 2D formula
 - projection to 2D preserves barycentric coordinates

Computing Triangle Area in 2D

- Suppose we project the triangle to xy plane
- $\text{Area}(\text{xy-projection}(ABC)) =$
$$(1/2) ((b_x - a_x)(c_y - a_y) - (c_x - a_x)(b_y - a_y))$$
- This formula gives correct sign
(important for barycentric coordinates)

Summary

- Ray-Surface Intersections
- Special cases: sphere, polygon
- Barycentric Coordinates