

Splines

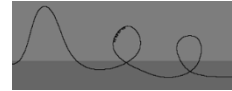
Hermite Splines
Bezier Splines
Catmull-Rom Splines
Other Cubic Splines
[Angel Ch. 10]

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Roller coaster

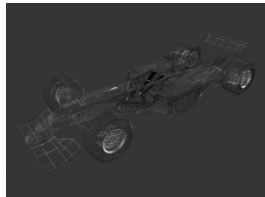
- Next programming assignment involves creating a 3D roller coaster animation
- We must model the 3D curve describing the roller coaster, but how?



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Modeling Complex Shapes

- We want to build models of very complicated objects
- Complexity is achieved using simple pieces
 - polygons,
 - parametric curves and surfaces, or
 - implicit curves and surfaces
- This lecture: **parametric curves**



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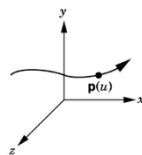
What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

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Curve Representations

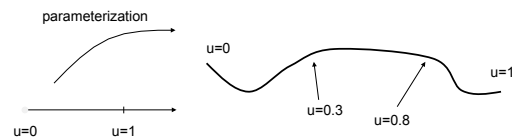
- **Explicit:** $y = f(x)$ $y = x^2$ $y = mx + b$
 - Must be a function (single-valued)
 - Big limitation—vertical lines?
- **Parametric:** $(x,y,z) = (f(u),g(u),h(u))$
 - + Easy to specify, modify, control
 - Extra “hidden” variable u , the *parameter*
 $(x,y) = (\cos u, \sin u)$
- **Implicit (2D):** $f(x,y) = 0$
 - + y can be a multiple valued function of x
 - Hard to specify, modify, control
 $x^2 + y^2 - r^2 = 0$



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Parameterization of a Curve

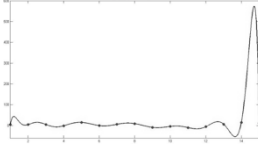
- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.
- Parameterization is not unique. It can be slow, fast, with continuous / discontinuous speed, clockwise (CW) or CCW...



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Polynomial Interpolation

- An n -th degree polynomial fits a curve to $n+1$ points
 - called Lagrange Interpolation
 - result is a curve that is too wiggly, change to any control point affects entire curve (non-local)
 - this method is poor
- We usually want the curve to be as smooth as possible
 - minimize the wiggles
 - high-degree polynomials are bad

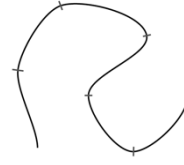


Lagrange interpolation, degree=15

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Splines: Piecewise Polynomials

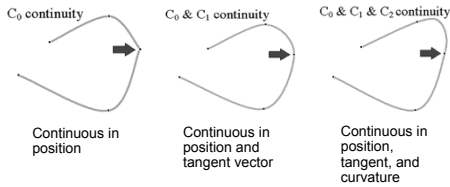
- A spline is a *piecewise polynomial*: Curve is broken into consecutive segments, each of which is a low-degree polynomial interpolating (passing through) the control points
- Cubic piecewise polynomials are the most common:
 - They are the lowest order polynomials that
 - interpolate two points and
 - allow the gradient at each point to be defined (C¹ continuity is possible).
 - Piecewise definition gives local control.
 - Higher or lower degrees are possible, of course.



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Piecewise Polynomials

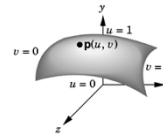
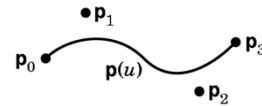
- Spline: many polynomials pieced together
- Want to make sure they fit together nicely



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Splines

- Types of splines:
 - Hermite Splines
 - Bezier Splines
 - Catmull-Rom Splines
 - Natural Cubic Splines
 - B-Splines
 - NURBS
- Splines can be used to model both curves and surfaces



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Cubic Curves in 3D

- Cubic polynomial:
 - $p(u) = au^3 + bu^2 + cu + d = [u^3 \ u^2 \ u \ 1] [a \ b \ c \ d]^T$
 - a, b, c, d are 3-vectors, u is a scalar
- Three cubic polynomials, one for each coordinate:
 - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
 - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
 - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$

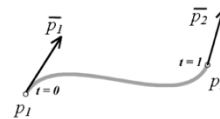
- In matrix notation:

$$[x(u) \ y(u) \ z(u)] = [u^3 \ u^2 \ u \ 1] \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

- Or simply: $p = [u^3 \ u^2 \ u \ 1] A$

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Cubic Hermite Splines



Hermite Specification

We want a way to specify the end points and the slope at the end points!

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Deriving Hermite Splines

- Four constraints: value and slope (in 3-D, position and tangent vector) at beginning and end of interval $[0,1]$:
 - $p(0) = p_1 = (x_1, y_1, z_1)$
 - $p(1) = p_2 = (x_2, y_2, z_2)$
 - $p'(0) = \bar{p}_1 = (\bar{x}_1, \bar{y}_1, \bar{z}_1)$
 - $p'(1) = \bar{p}_2 = (\bar{x}_2, \bar{y}_2, \bar{z}_2)$
 ← the user constraints
- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$
- Four unknowns: a, b, c, d

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Deriving Hermite Splines

- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$
 - $p_1 = p(0) = d$
 - $p_2 = p(1) = a + b + c + d$
 - $\bar{p}_1 = p'(0) = c$
 - $\bar{p}_2 = p'(1) = 3a + 2b + c$
- Linear system: 12 equations for 12 unknowns (however, can be simplified to 4 equations for 4 unknowns)
- Unknowns: a, b, c, d (each of a, b, c, d is a 3-vector)

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Deriving Hermite Splines

$$\begin{aligned} d &= p_1 \\ a + b + c + d &= p_2 \\ c &= \bar{p}_1 \\ 3a + 2b + c &= \bar{p}_2 \end{aligned}$$

Rewrite this 12x12 system as a 4x4 system:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \end{bmatrix}$$

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The Cubic Hermite Spline Equation

- After inverting the 4x4 matrix, we obtain:

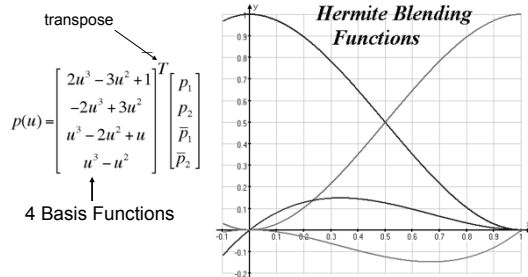
$$[x \ y \ z] = [u^3 \ u^2 \ u \ 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \end{bmatrix}$$

↑ point on the spline
↑ parameter vector
basis
control matrix (what the user gets to pick)

- This form is typical for splines
 - basis matrix and meaning of control matrix change with the spline type

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Four Basis Functions for Hermite Splines



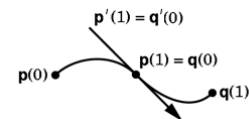
Every cubic Hermite spline is a linear combination (blend) of these 4 functions.

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Piecing together Hermite Splines

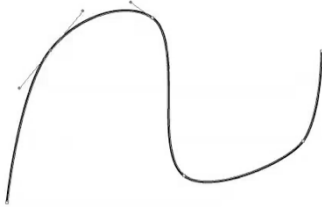
It's easy to make a multi-segment Hermite spline:

- each segment is specified by a cubic Hermite curve
- just specify the position and tangent at each "joint" (called *knot*)
- the pieces fit together with matched positions and first derivatives
- gives C1 continuity



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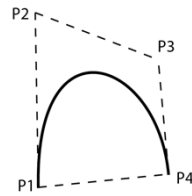
Hermite Splines in Adobe Illustrator



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Bezier Splines

- Variant of the Hermite spline
- Instead of endpoints and tangents, four control points
 - points P1 and P4 are on the curve
 - points P2 and P3 are off the curve
 - $p(0) = P1$, $p(1) = P4$,
 - $p'(0) = 3(P2-P1)$, $p'(1) = 3(P4 - P3)$
- Basis matrix is derived from the Hermite basis (or from scratch)
- Convex Hull property: curve contained within the convex hull of control points
- Scale factor "3" is chosen to make "velocity" approximately constant



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The Bezier Spline Matrix

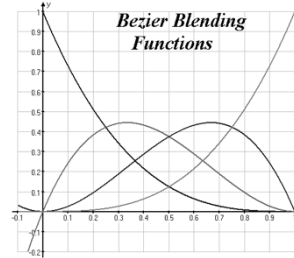
$$\begin{aligned}
 [x \ y \ z] &= [u^3 \ u^2 \ u \ 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} \\
 &= [u^3 \ u^2 \ u \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}
 \end{aligned}$$

Hermite basis
Bezier to Hermite
Bezier control matrix

Bezier basis
Bezier control matrix

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Bezier Blending Functions

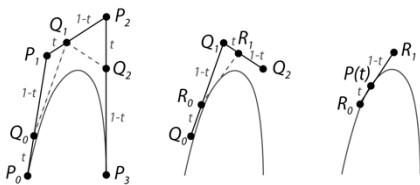


$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^T \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Also known as the order 4, degree 3 Bernstein polynomials
 Nonnegative, sum to 1
 The entire curve lies inside the polyhedron bounded by the control points

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DeCasteljau Construction

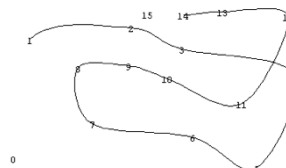


Efficient algorithm to evaluate Bezier splines.
 Similar to Horner rule for polynomials.
 Can be extended to interpolations of 3D rotations.

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Catmull-Rom Splines

- Roller-coaster (next programming assignment)
- With Hermite splines, the designer must arrange for consecutive tangents to be collinear, to get C^1 continuity. Similar for Bezier. This gets tedious.
- Catmull-Rom: an interpolating cubic spline with *built-in* C^1 continuity.
- Compared to Hermite/Bezier: fewer control points required, but less freedom.



Catmull-Rom spline

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Constructing the Catmull-Rom Spline

Suppose we are given n control points in 3-D: p_1, p_2, \dots, p_n .

For a Catmull-Rom spline, we set the tangent at p_i to $s * (p_{i+1} - p_{i-1})$ for $i=2, \dots, n-1$, for some s (often $s=0.5$)

s is *tension parameter*: determines the magnitude (but not direction!) of the tangent vector at point p_i

What about endpoint tangents? Use extra control points p_0, p_{n+1} .

Now we have positions and tangents at each knot. This is a Hermite specification. Now, just use Hermite formulas to derive the spline.

Note: curve between p_i and p_{i+1} is completely determined by $p_{i-1}, p_i, p_{i+1}, p_{i+2}$.

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Catmull-Rom Spline Matrix

$$[x \ y \ z] = [u^3 \ u^2 \ u \ 1] \begin{matrix} \begin{matrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{matrix} & \begin{matrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} \\ \text{basis} & \text{control matrix} \end{matrix} \end{matrix}$$

- Derived in way similar to Hermite and Bezier
- Parameter s is typically set to $s=1/2$.

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Splines with More Continuity?

- So far, only C^1 continuity.
- How could we get C^2 continuity at control points?

Possible answers:

- Use higher degree polynomials
degree 4 = quartic, degree 5 = quintic, ... but these get computationally expensive, and sometimes wiggly
- Give up local control → natural cubic splines
A change to any control point affects the entire curve
- Give up interpolation → cubic B-splines
Curve goes near, but not through, the control points

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Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

Summary:

Cannot get C^2 , interpolation and local control with cubics

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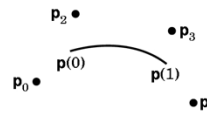
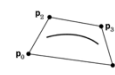
Natural Cubic Splines

- If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines*
- It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)
- Finding all the right weights is a *global* calculation (solve tridiagonal linear system)

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B-Splines

- Give up interpolation
 - the curve passes *near* the control points
 - best generated with interactive placement (because it's hard to guess where the curve will go)
- Curve obeys the convex hull property
- C^2 continuity and local control are good compensation for loss of interpolation



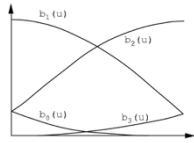
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B-Spline Basis

- We always need 3 more control points than the number of spline segments

$$M_{Bs} = \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bs} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



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Other Common Types of Splines

- Non-uniform Splines
- Non-Uniform Rational Cubic curves (NURBS)
- NURBS are very popular and used in many commercial packages

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How to Draw Spline Curves

- Basis matrix equation allows same code to draw any spline type
- Method 1:** brute force
 - Calculate the coefficients
 - For each cubic segment, vary u from 0 to 1 (fixed step size)
 - Plug in u value, matrix multiply to compute position on curve
 - Draw line segment from last position to current position
- What's wrong with this approach?
 - Draws in even steps of u
 - Even steps of u does not mean even steps of x
 - Line length will vary over the curve
 - Want to bound line length
 - too long: curve looks jagged
 - too short: curve is slow to draw

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Drawing Splines, 2

- Method 2:** recursive subdivision - vary step size to draw short lines

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = F(u0)
    x1 = F(u1)
    if |x1 - x0| > maxlinelength
        Subdivide(u0,umid,maxlinelength)
        Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- Variant on Method 2** - subdivide based on curvature
 - replace condition in "if" statement with straightness criterion
 - draws fewer lines in flatter regions of the curve

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Summary

- Piecewise cubic is generally sufficient
- Define conditions on the curves and their continuity
- Most important:
 - basic curve properties (what are the conditions, controls, and properties for each spline type)
 - generic matrix formula for uniform cubic splines $p(u) = u B G$
 - given a definition, derive a basis matrix (do not memorize the matrices themselves)

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