CSCI 420 Computer Graphics
Lecture 5

Transformations

Vector Spaces
Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 3]

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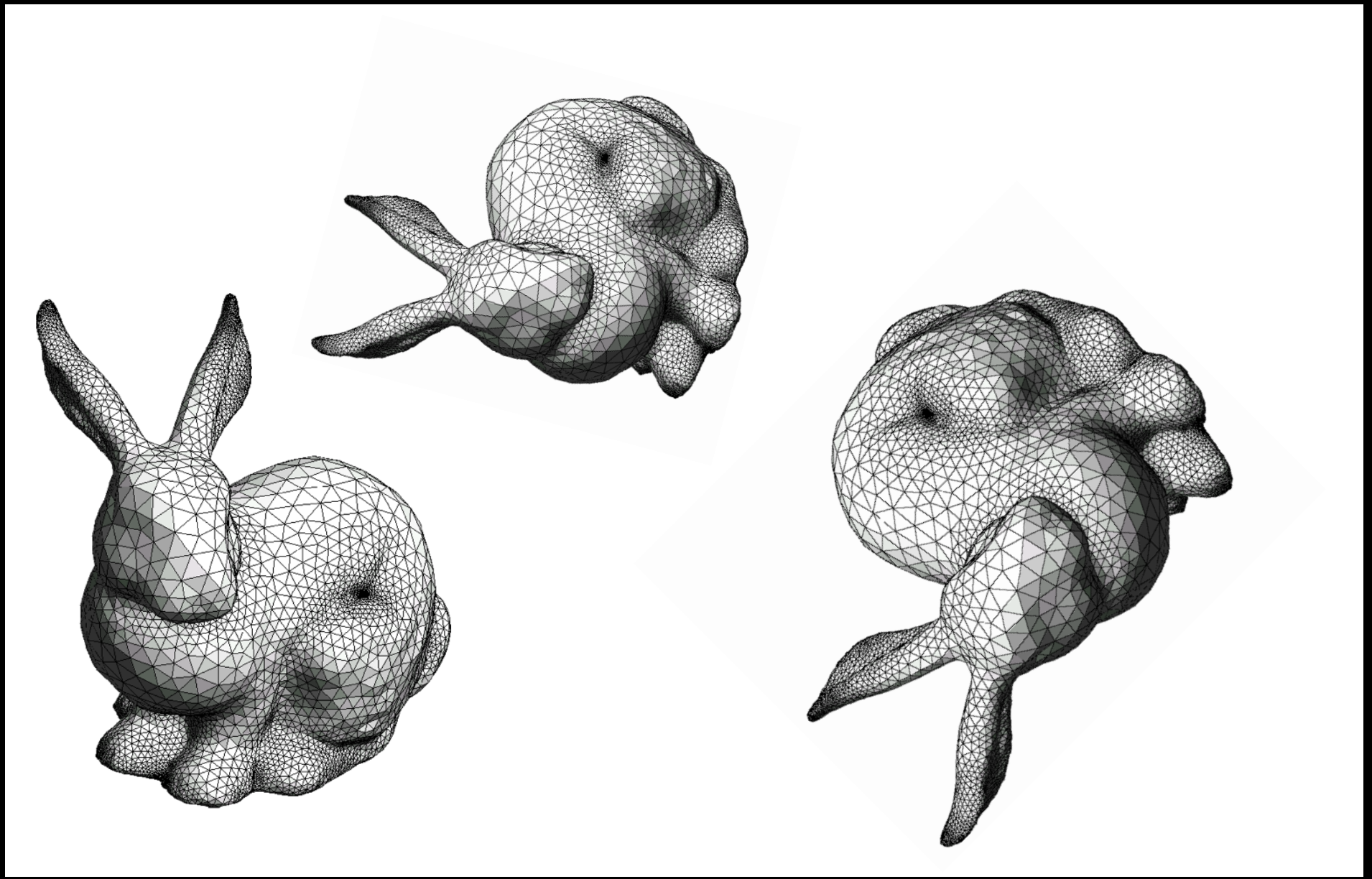
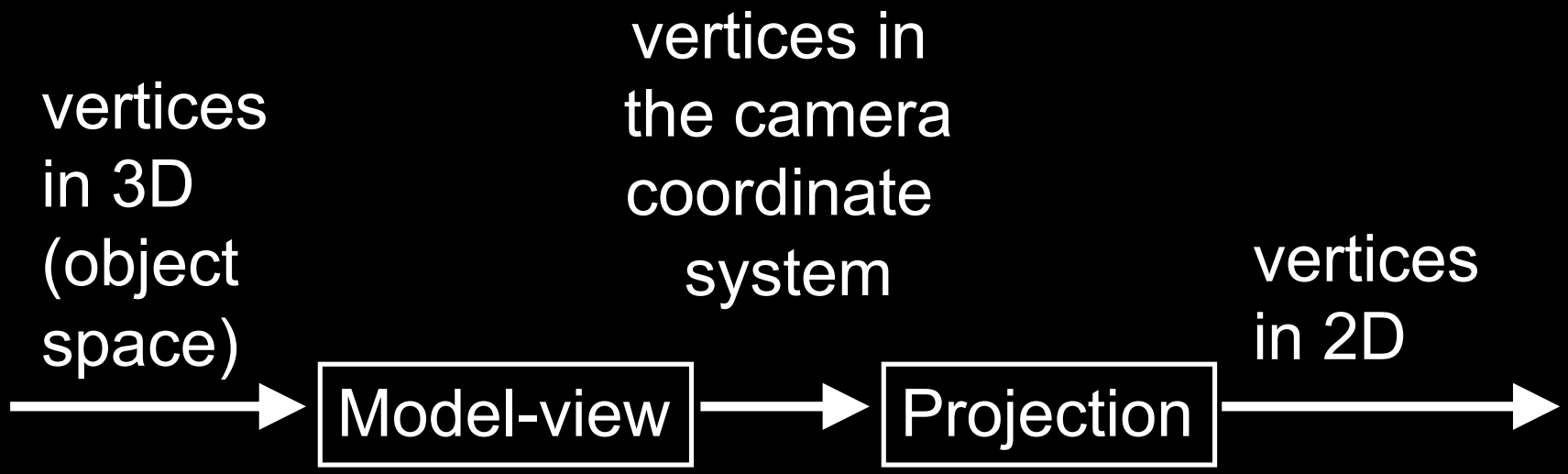
OpenGL Transformations



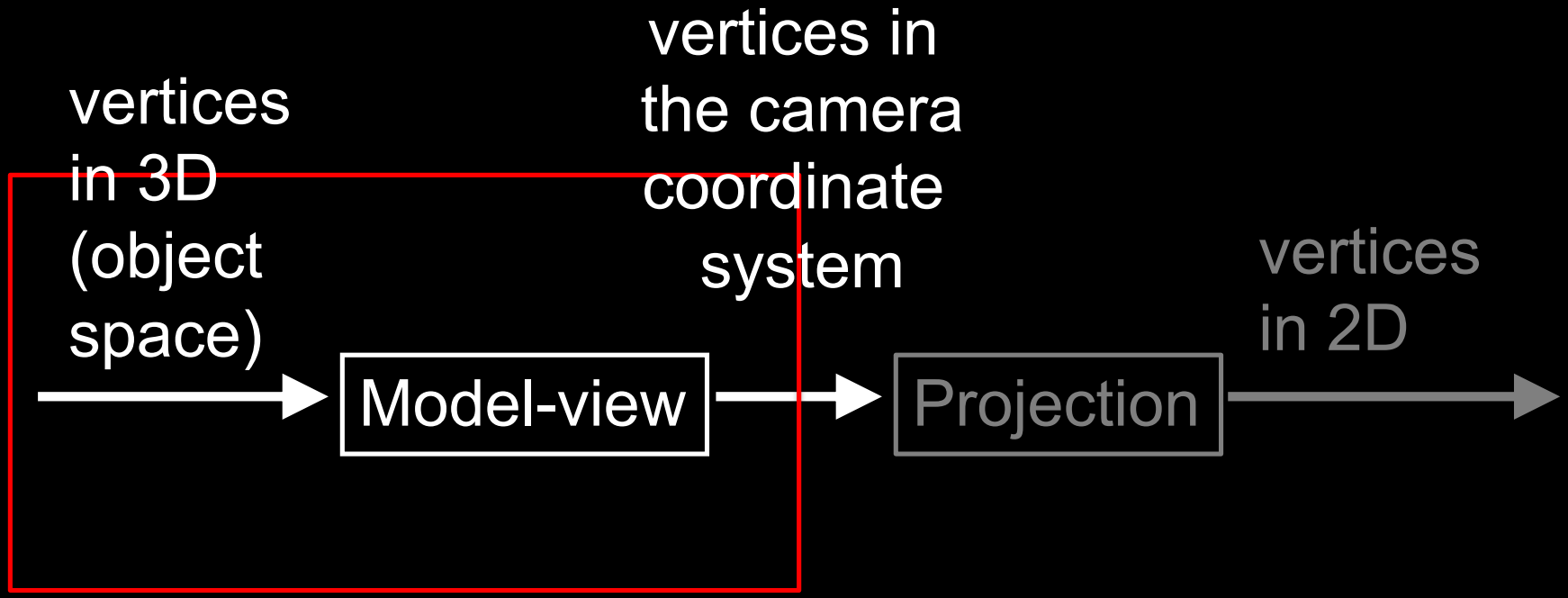
OpenGL Transformation Matrices

- Model-view matrix (4x4 matrix)
- Projection matrix (4x4 matrix)



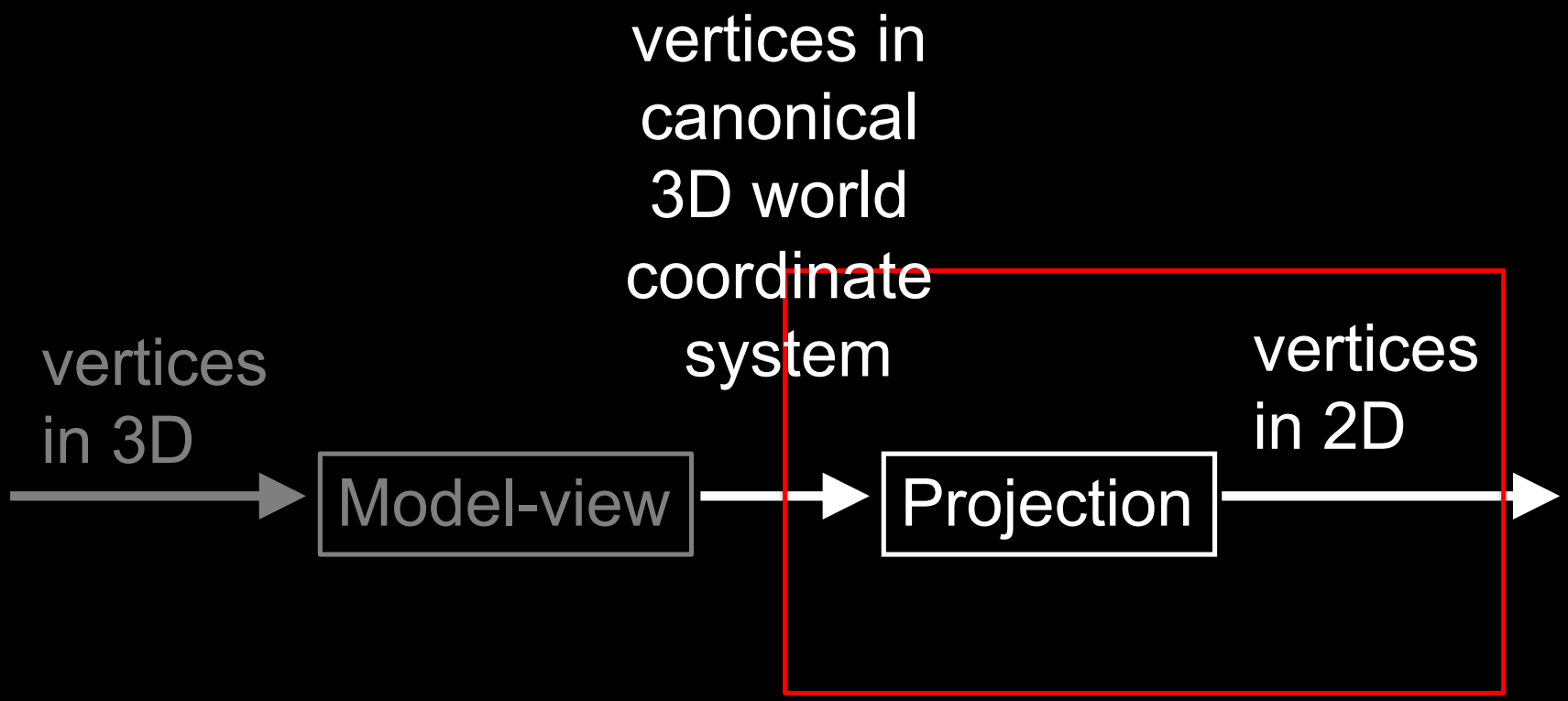
4x4 Model-view Matrix (this lecture)

- Translate, rotate, scale objects
- Position the camera



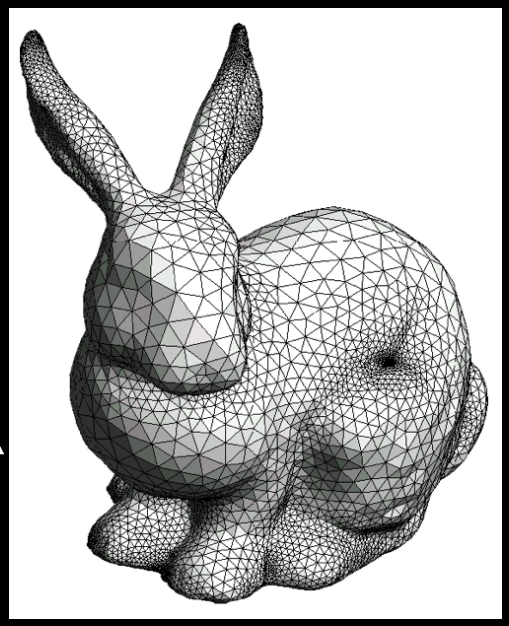
4x4 Projection Matrix (next lecture)

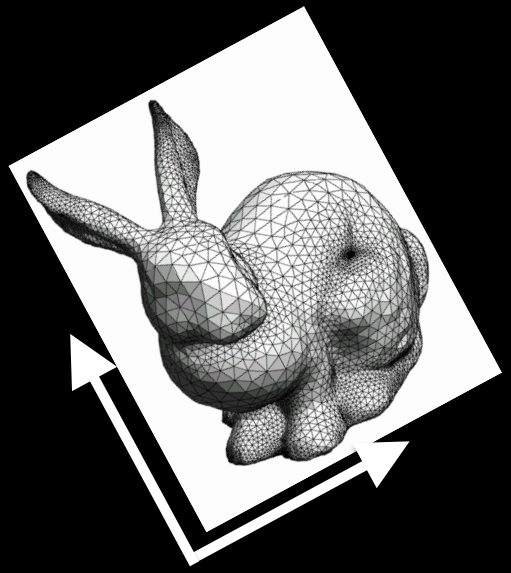
- Project from 3D to 2D

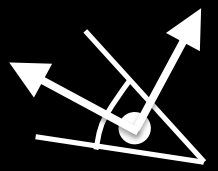


4x4 Model-view Matrix (this lecture)

- Translate, rotate, scale objects in world space
- Position and orient the camera





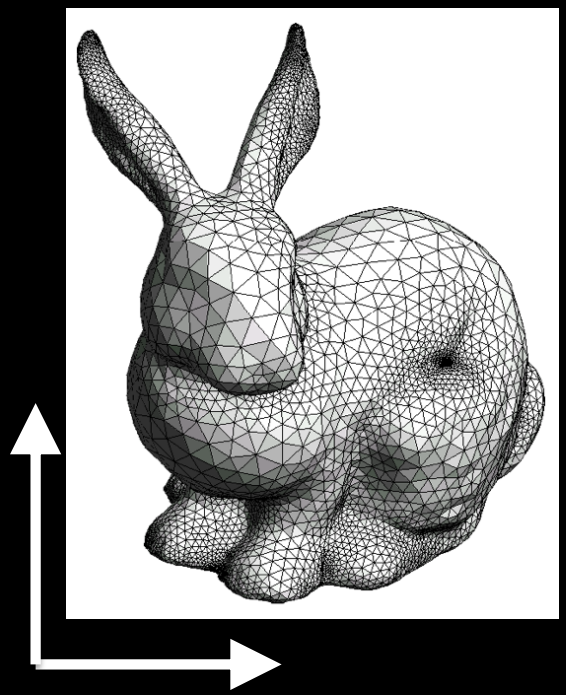


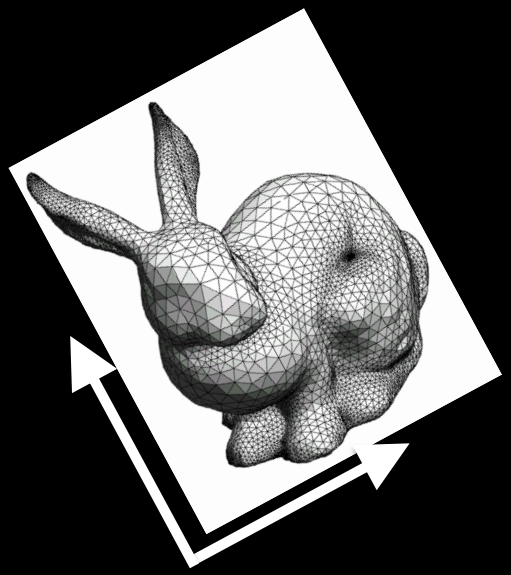
camera

world

4x4 Model Matrix

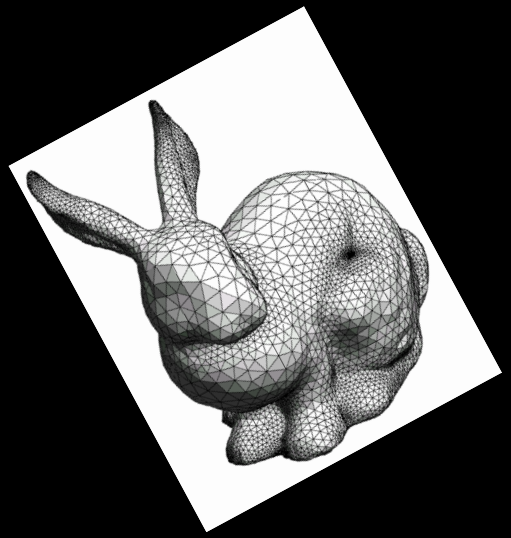
- Translate, rotate, scale objects in world space

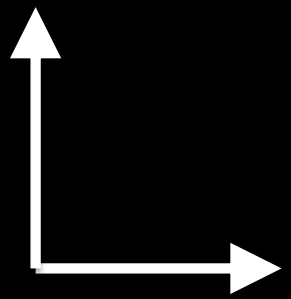




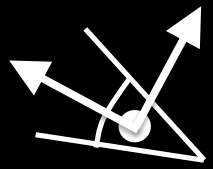
4x4 View Matrix

- Position and orient the camera
- From world space to camera space



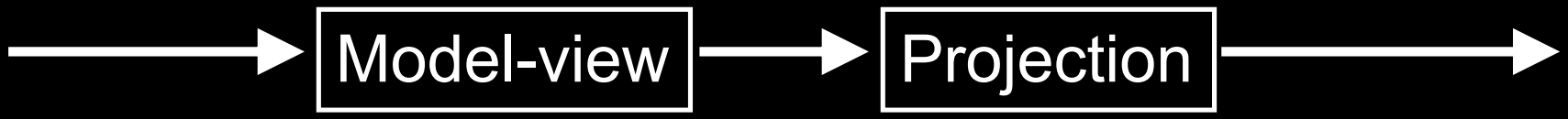


world



camera

OpenGL Transformation Matrices



- Manipulated separately in OpenGL
- Core profile: set them directly
- Compatibility profile: must set matrix mode

```
glMatrixMode (GL_MODELVIEW);
glMatrixMode (GL_PROJECTION);
```

Setting the Modelview Matrix: Core Profile

- Set identity:

```
openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);
openGLMatrix.LoadIdentity();
```

- Use our openGLMatrix library functions:

```
openGLMatrix.Translate(dx, dy, dz);
openGLMatrix.Rotate(angle, vx, vy, vz);
openGLMatrix.Scale(sx, sy, sz);
```

- Upload m to the GPU:

```
float m[16]; // column-major
openGLMatrix.GetMatrix(m);
GLboolean isRowMajor = GL_FALSE;
glUniformMatrix4fv(h_modelViewMatrix, 1, isRowMajor, m);
// note: h_modelViewMatrix is a handle of the shader
   modelview matrix variable (will discuss in Shaders lecture)
```

Setting the Modelview Matrix: Compatibility Profile

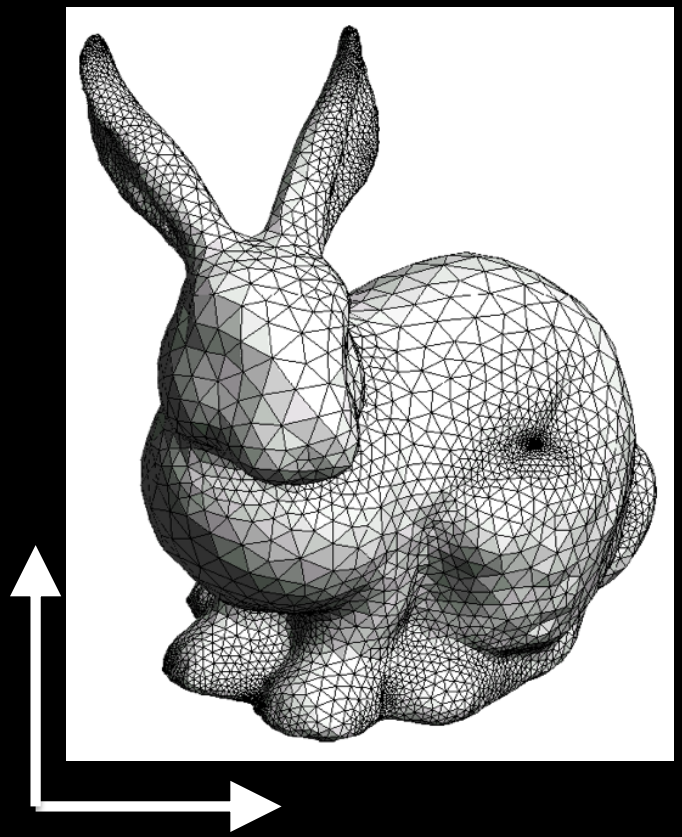
- Load or post-multiply

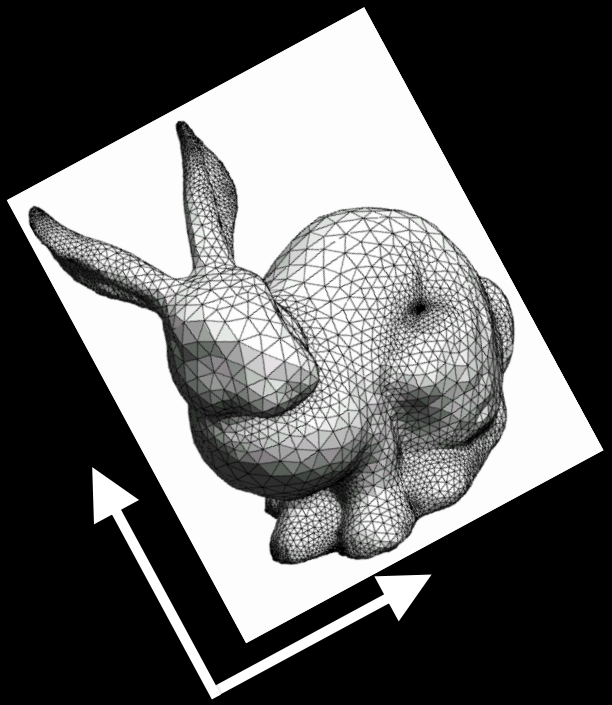
```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity(); // very common usage
float m[16] = { … };
glLoadMatrixf(m); // rare, advanced
glMultMatrixf(m); // rare, advanced
```

- Use library functions

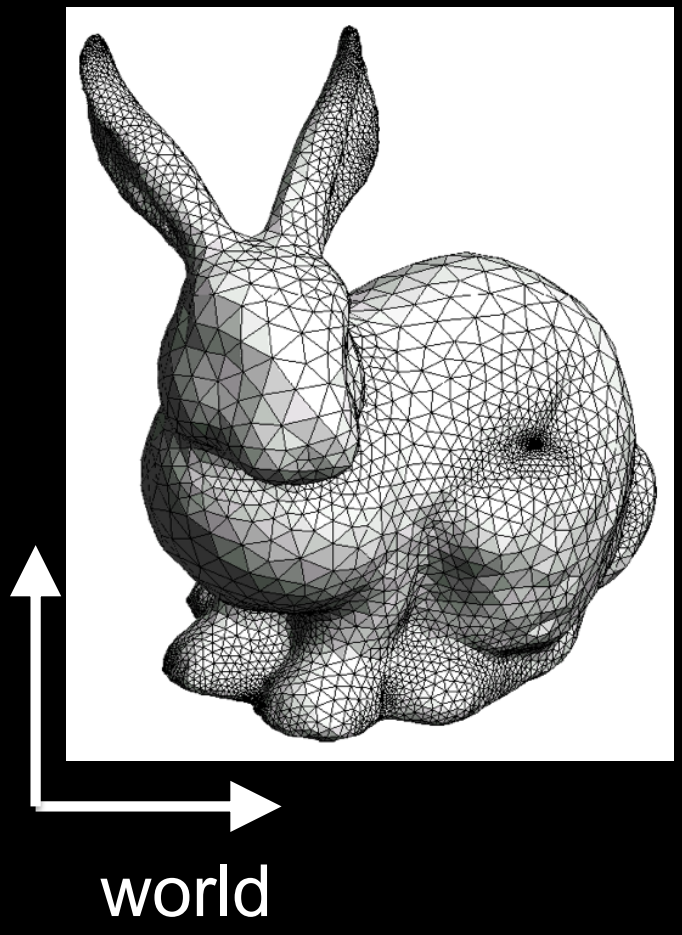
```
glTranslatef(dx, dy, dz);
glRotatef(angle, vx, vy, vz);
glScalef(sx, sy, sz);
```

Translated, rotated, scaled object





The *rendering* coordinate system

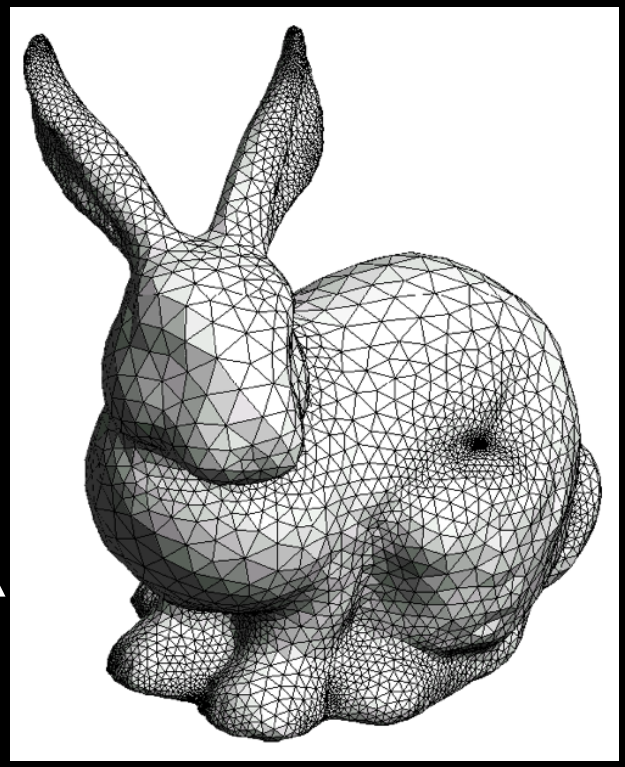


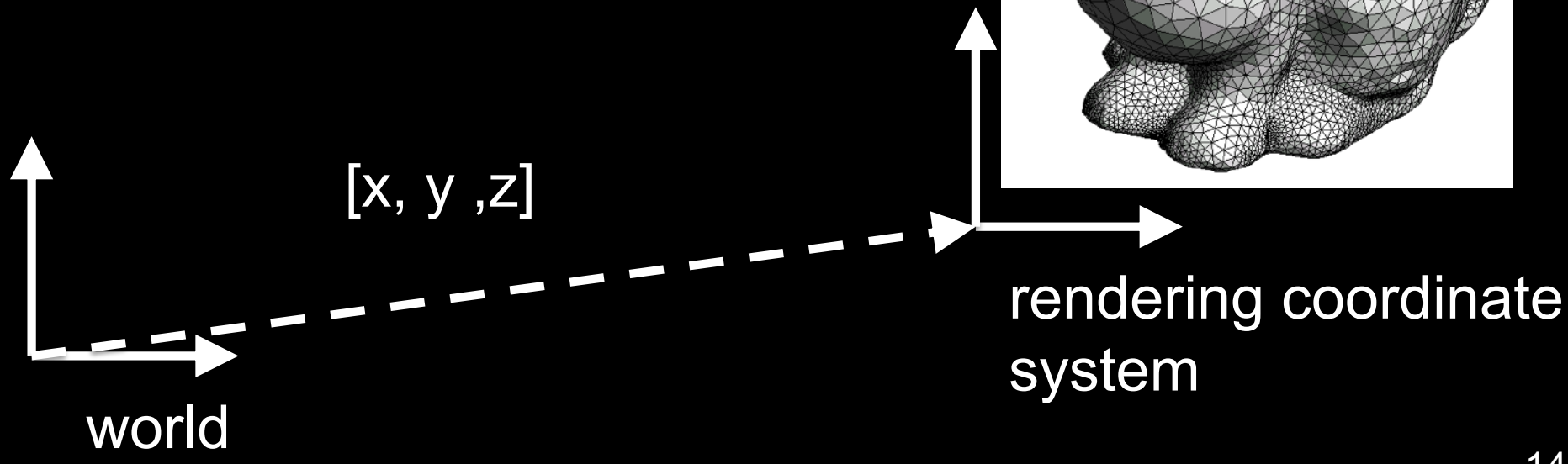
Initially (after LoadIdentity()) :

rendering coordinate system = world coordinate system

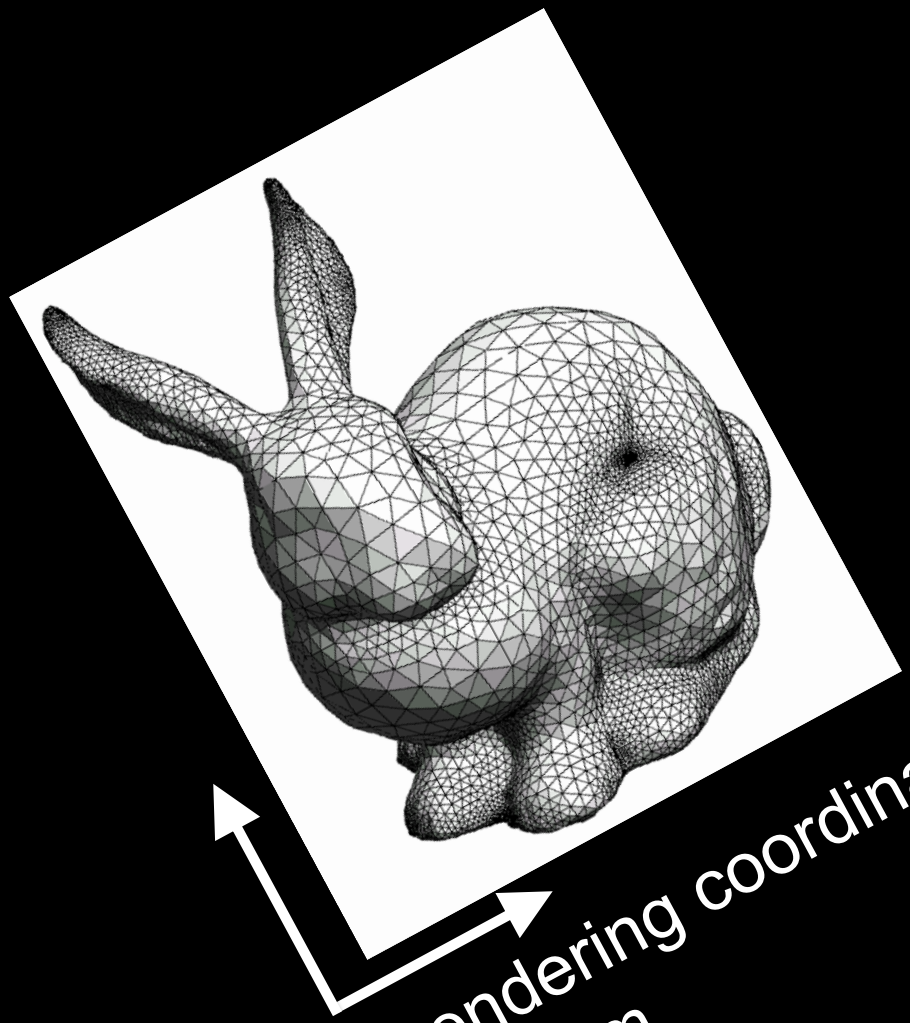
The *rendering* coordinate system

Translate(x, y, z);

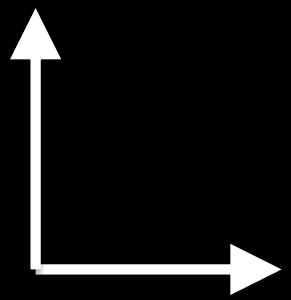




The *rendering* coordinate system

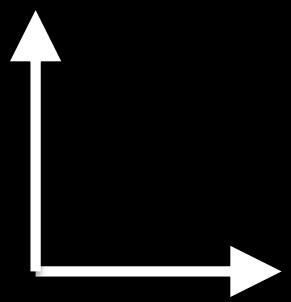


Rotate(angle, ax, ay, az);

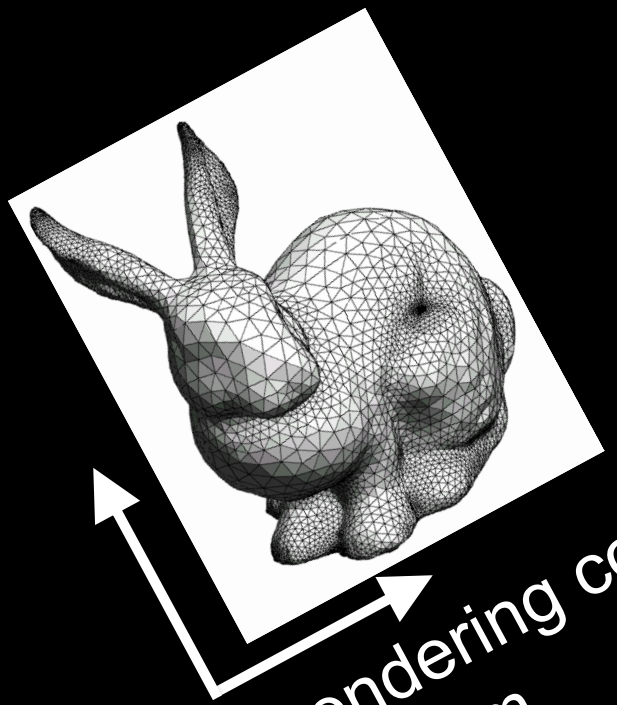


The *rendering* coordinate system

Scale(sx, sy, sz);



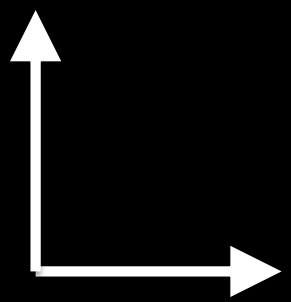
world

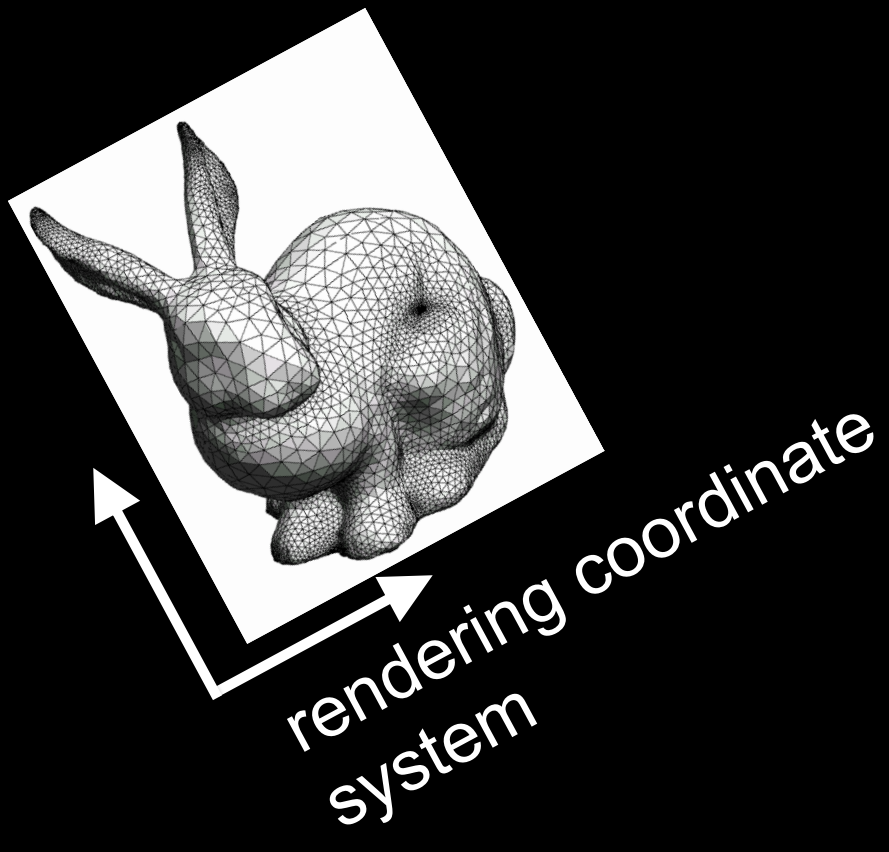


rendering coordinate system

OpenGL pseudo-code

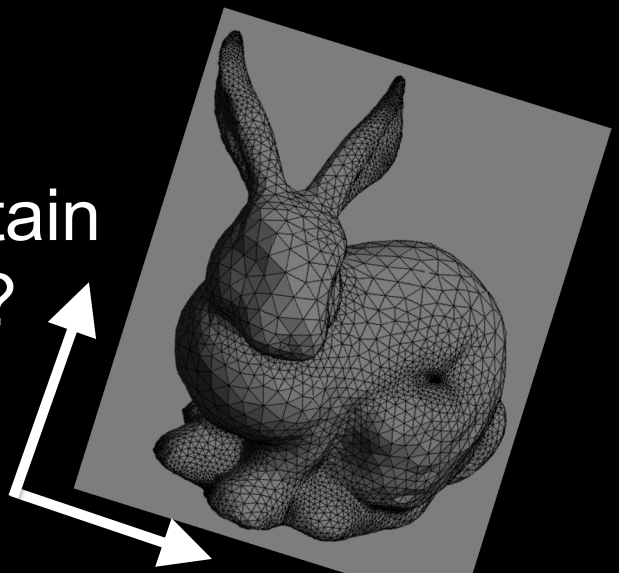
```
MatrixMode(ModelView);
LoadIdentity();
Translate(x, y, z);
Rotate(angle, ax, ay, az);
Scale(sx, sy, sz);
glUniformMatrix4fv(…);
renderBunny();
```

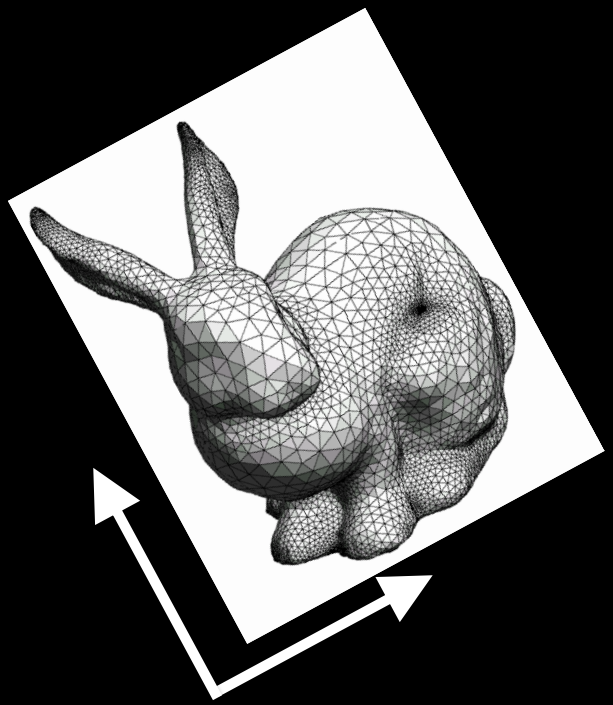


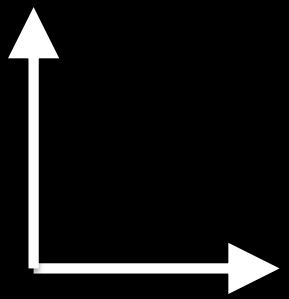


Rendering more objects

How to obtain this frame?





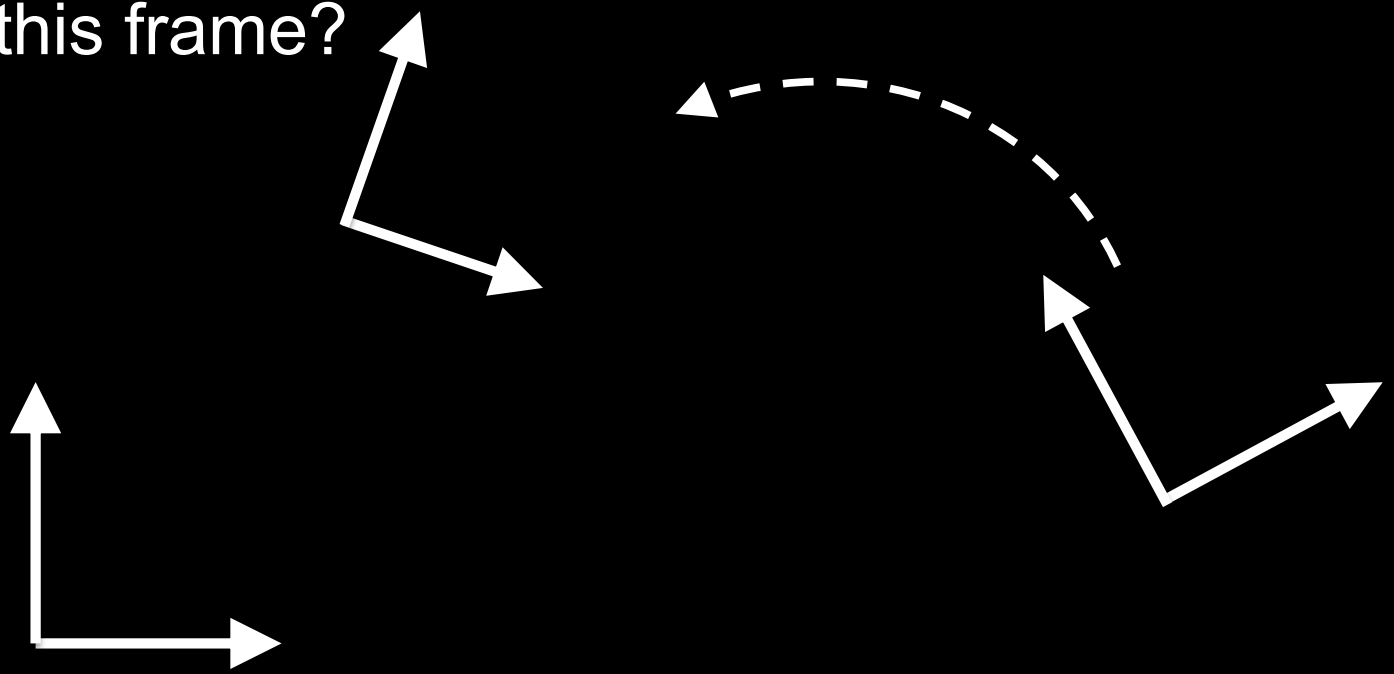


Rendering more objects

Solution 1:

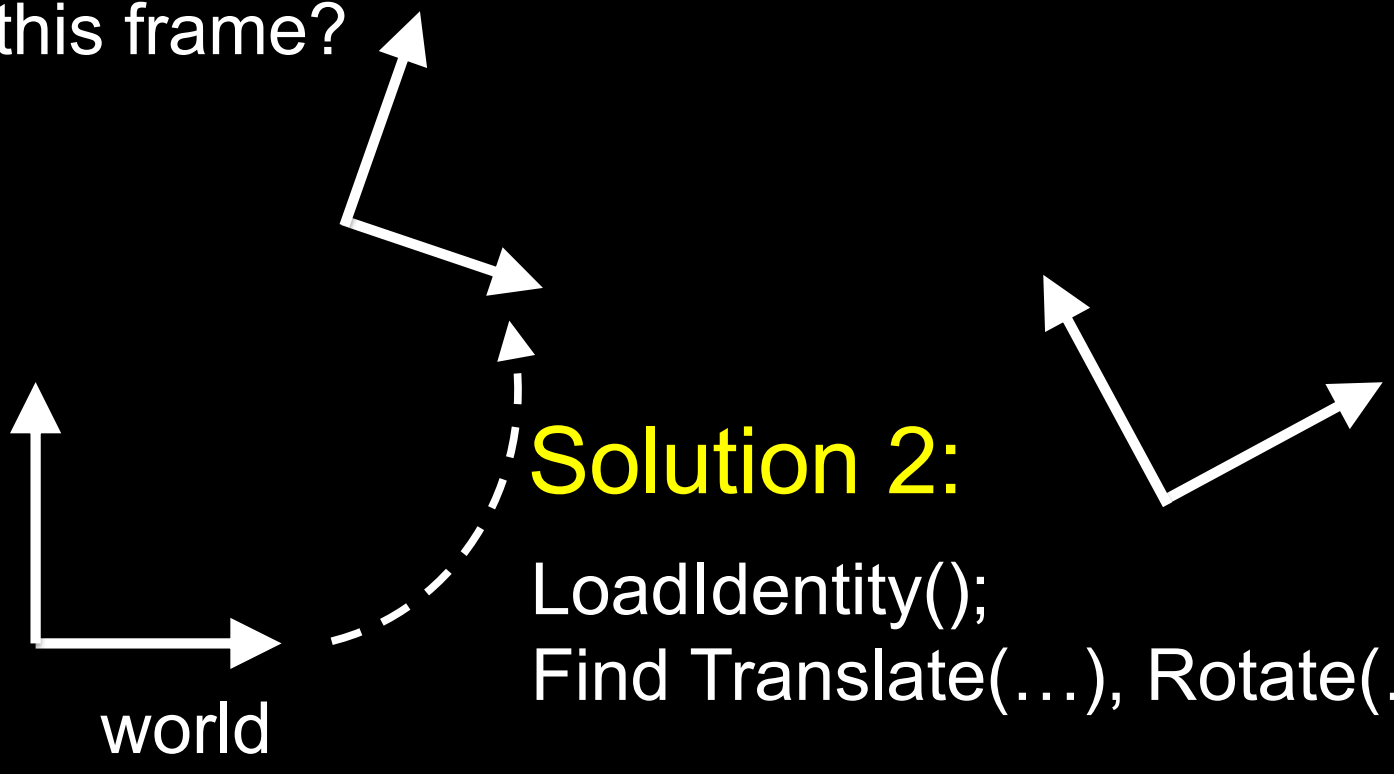
Find Translate(…), Rotate(…),
Scale(…)

How to obtain
this frame?



Rendering more objects

How to obtain this frame?



Solution 2:

LoadIdentity();
Find Translate(…), Rotate(…), Scale(…)

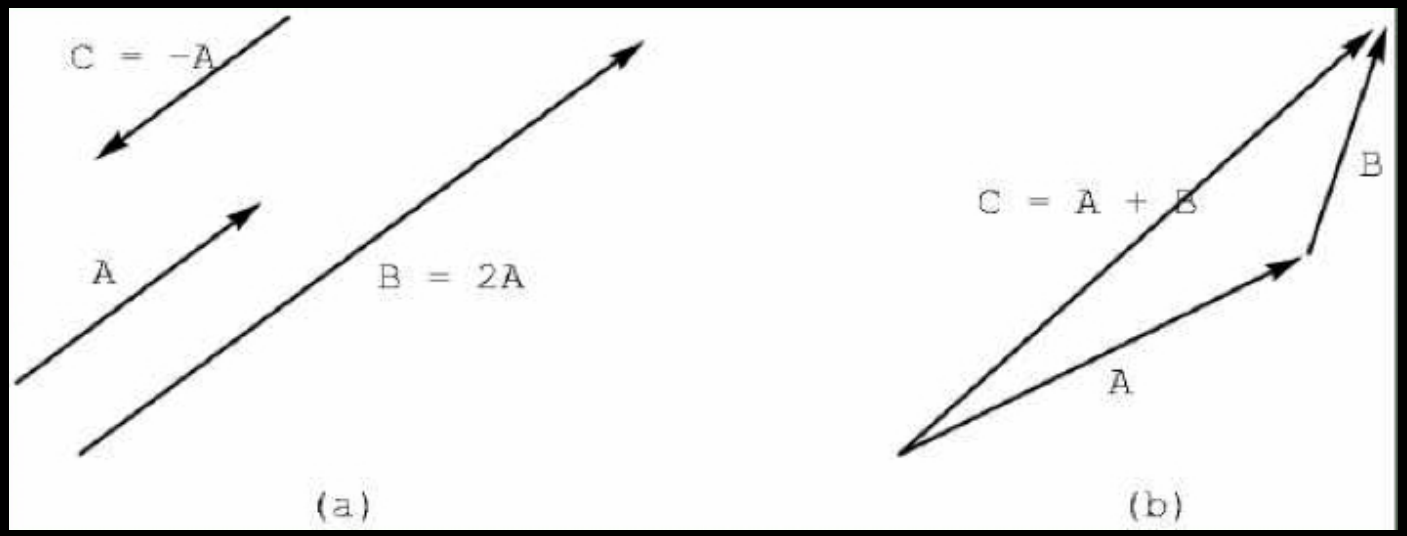
3D Math Review

Scalars

- Scalars α, β, γ from a *scalar field*
- Operations $\alpha+\beta$, $\alpha \cdot \beta$, 0, 1, $-\alpha$, $(\ )^{-1}$
- “Expected” laws apply
- Examples: rationals or reals with addition and multiplication

Vectors

- Vectors *u, v, w* from a *vector space*
- Vector addition *u* + *v* , subtraction *u* - *v*
- Zero vector **0**
- Scalar multiplication α *v*

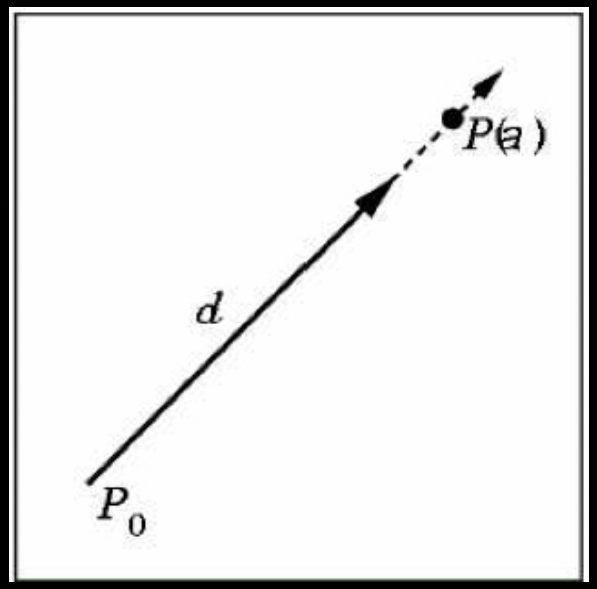


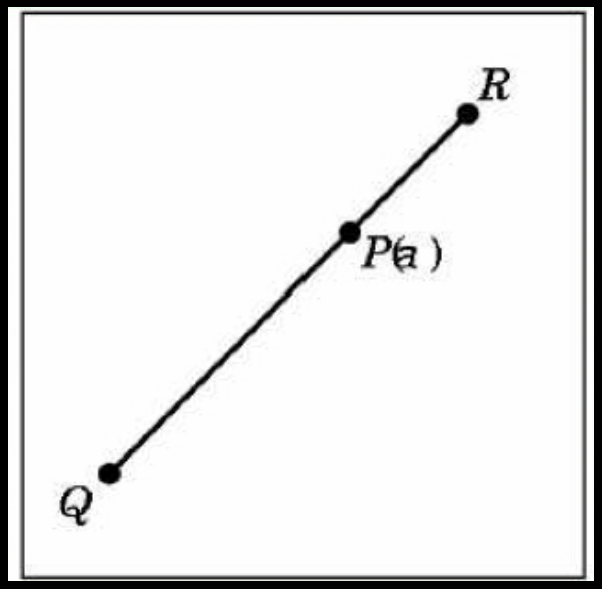
Euclidean Space

- Vector space over real numbers
- Three-dimensional in computer graphics
- Dot product: $\alpha = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- *u, v* are *orthogonal* if $u \cdot v = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of *v*

Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha\, d$



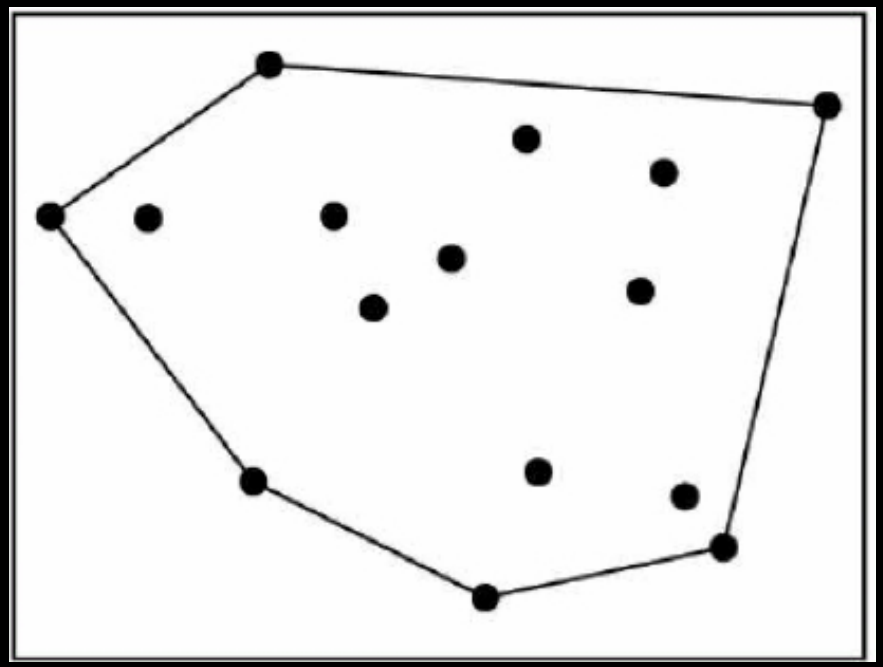


- Line segment between Q and R:
  $P(\alpha) = (1-\alpha)\, Q + \alpha\, R$ for $0 \le \alpha \le 1$

Convex Hull

- Convex hull defined by

$$P = \alpha_1 P_1 + \ldots + \alpha_n P_n$$
$$\text{for } \alpha_1 + \ldots + \alpha_n = 1$$
$$\text{and } 0 \le \alpha_i \le 1, \; i = 1, \ldots, n$$

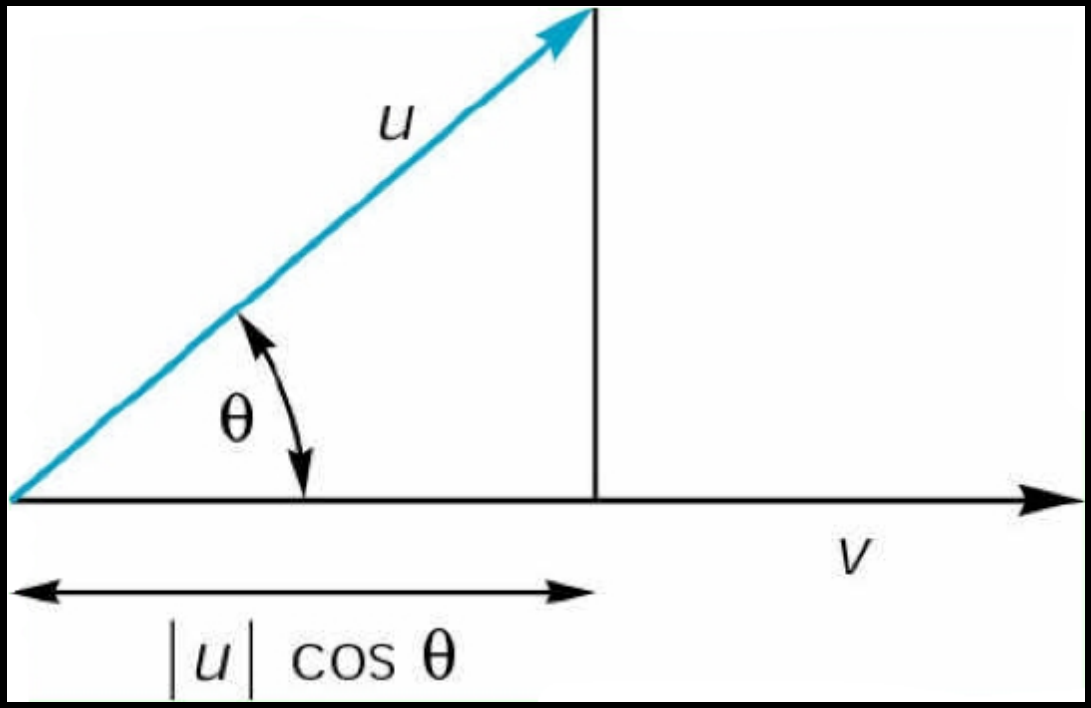


Projection

- Dot product projects one vector onto another vector

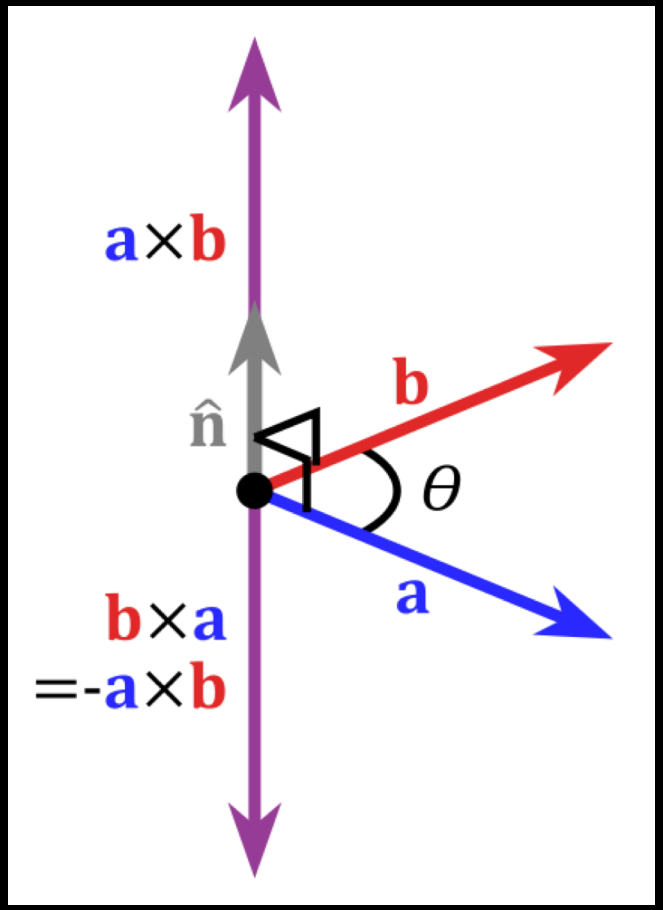
$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\,|v| \cos(\theta)$$

$$\mathrm{pr}_v\, u = (u \cdot v)\, v / |v|^2$$

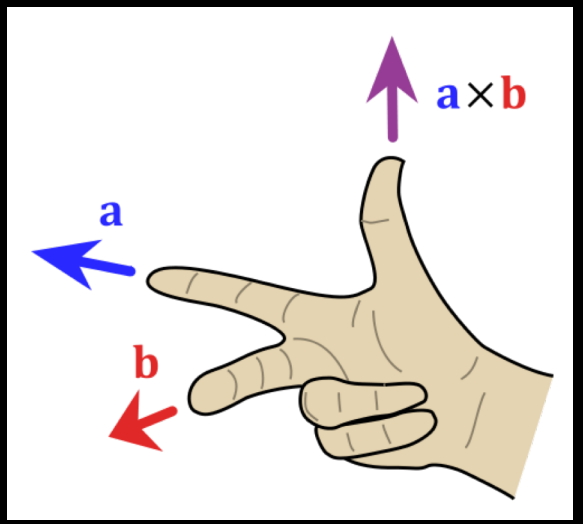


Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$



- |a x b| = |a| |b| |sin(θ)|
- Cross product is perpendicular to both a and b
- Right-hand rule



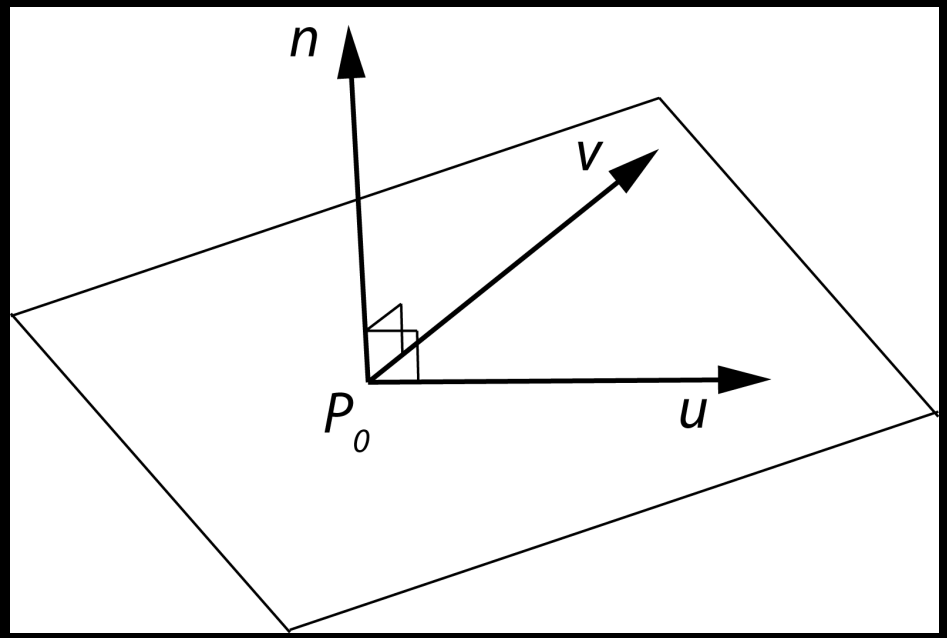
Source: Wikipedia

Plane

- Plane defined by point P_0 and vectors u and v
- u and v should not be parallel
- Parametric form:

  $T(\alpha, \beta) = P_0 + \alpha u + \beta v$

  (α and β are scalars)



- $n = u \times v / |u \times v|$ is the normal
- $n \cdot (P - P_0) = 0$ if and only if P lies in plane

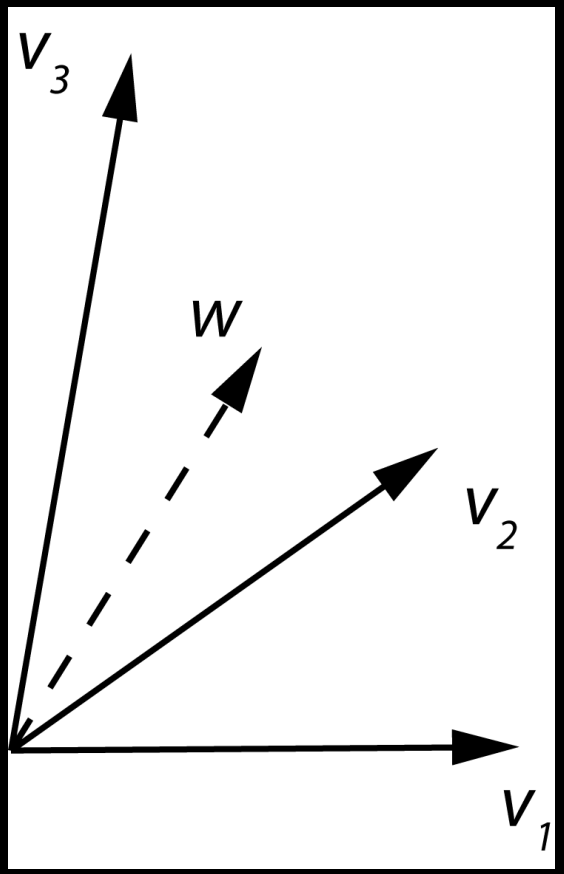
Coordinate Systems

- Let v_1, v_2, v_3 be three linearly independent vectors in a 3-dimensional vector space

- Can write *any* vector w as

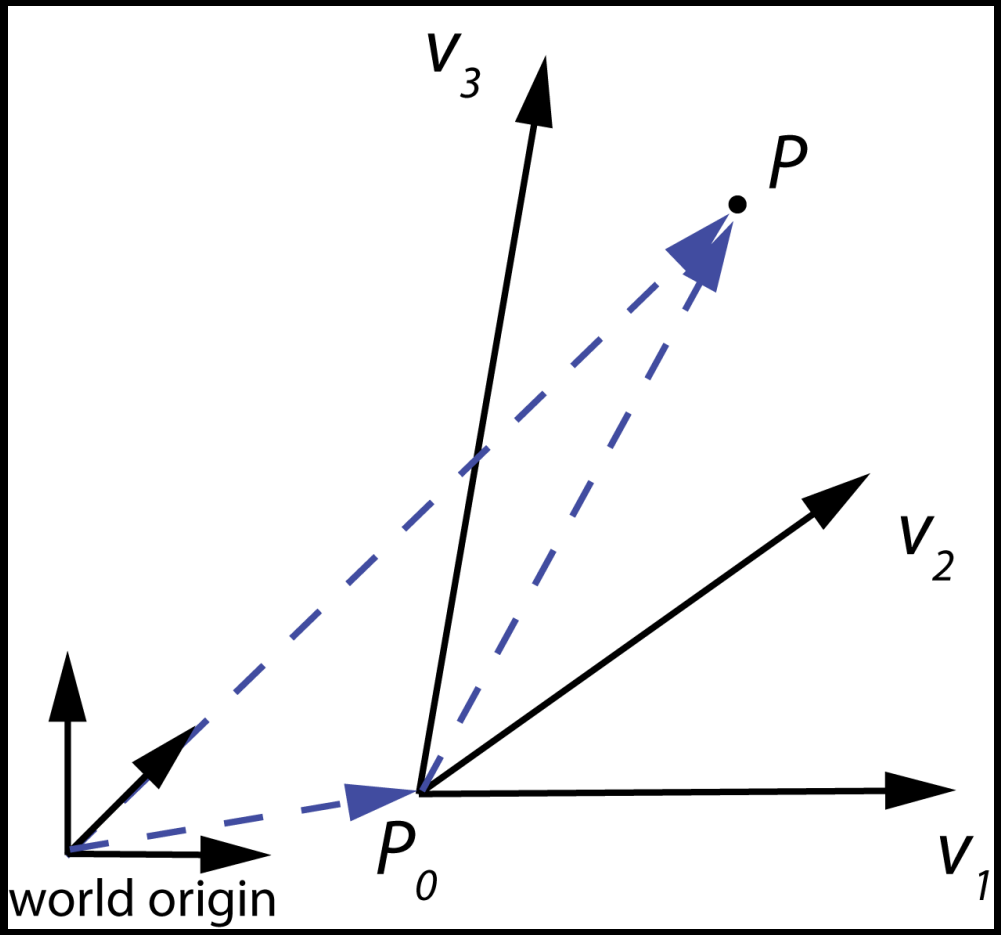
$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

for some scalars α_1, α_2, α_3

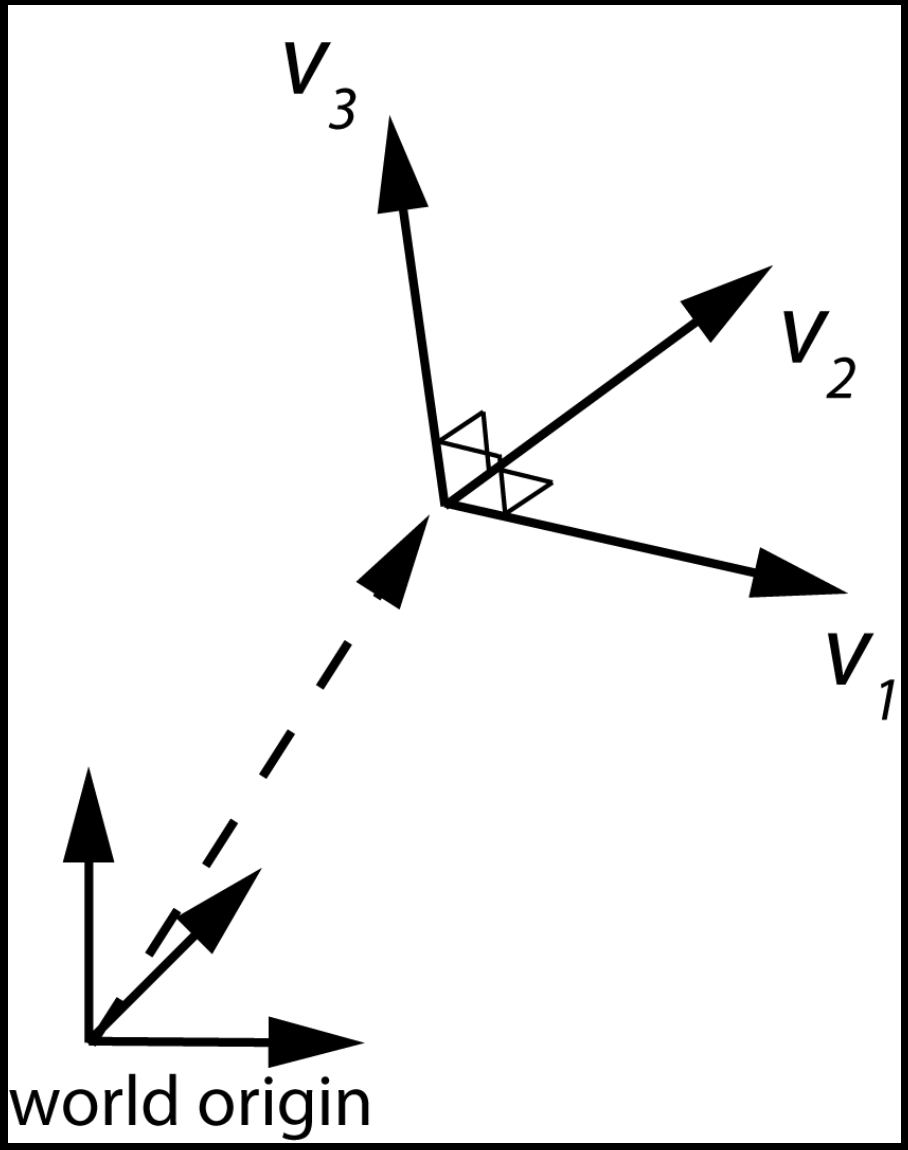


Frames

- Frame = origin P_0 + coordinate system
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$



In Practice, Frames are Often Orthogonal



Representing 3D transformations (and model-view matrices)

Linear Transformations

- 3 x 3 matrices represent linear transformations
  a = **Mb**
- Can represent rotation, scaling, and reflection
- Cannot represent translation

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

In order to represent rotations, scales AND translations: Homogeneous Coordinates

- Augment $[\alpha_1 \;\; \alpha_2 \;\; \alpha_3]^T$ by adding a fourth component (1):

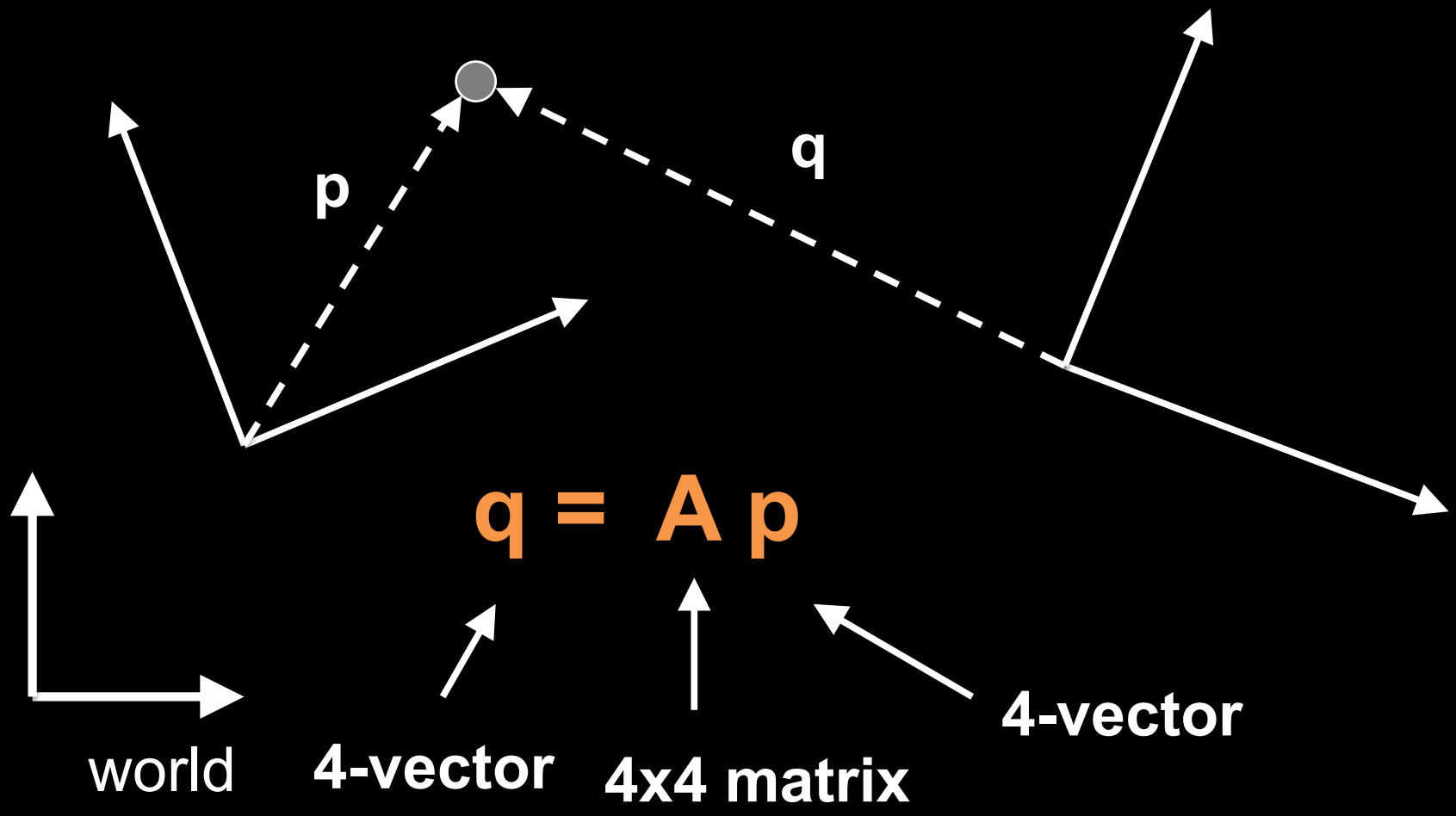
  $\mathbf{p} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T$

- Homogeneous property:

  $\mathbf{p} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T = [\beta\alpha_1 \;\; \beta\alpha_2 \;\; \beta\alpha_3 \;\; \beta]^T$,

  for any scalar $\beta \neq 0$

Homogeneous coordinates are transformed by 4x4 matrices



Affine Transformations (4x4 matrices)

- Translation
- Rotation
- Scaling
- Any composition of the above
- Later: projective (perspective) transformations
  - Also expressible as 4 x 4 matrices!

Translation

- **q** = **p** + **d** where **d** = $[\alpha_x \;\; \alpha_y \;\; \alpha_z \;\; 0]^T$
- **p** = $[x \;\; y \;\; z \;\; 1]^T$
- **q** = $[x' \;\; y' \;\; z' \;\; 1]^T$
- Express in matrix form **q** = **T p** and solve for **T**

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- $x' = \beta_x x$
- $y' = \beta_y y$
- $z' = \beta_z z$
- Express as **q** = **S p** and solve for **S**

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation in 2 Dimensions

- Rotation by θ about the origin
- x' = x cos θ – y sin θ
- y' = x sin θ + y cos θ

- Express in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note that the determinant is 1

Rotation in 3 Dimensions

- Orthogonal matrices:

$$RR^T = R^TR = I$$
$$\det(R) = 1$$

- Affine transformation:

$$A = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Affine Matrices are Composed by Matrix Multiplication

- $\mathbf{A} = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3$
- Applied from right to left
- $\mathbf{A}\,\mathbf{p} = (\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3)\,\mathbf{p} = \mathbf{A}_1 (\mathbf{A}_2 (\mathbf{A}_3\, p))$
- Compatibility mode:
  When calling glTranslate3f, glRotatef, or glScalef,
  OpenGL forms the corresponding 4x4 matrix,
  and multiplies the current modelview matrix with it.

Summary

- OpenGL Transformation Matrices
- Vector Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices