CSCI 420 Computer Graphics
Lecture 6

Viewing and Projection

Shear Transformation
Camera Positioning
Simple Parallel Projections
Simple Perspective Projections
[Angel, Ch. 4]

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Reminder: Affine Transformations

- Given a point [x y z], form homogeneous coordinates [x y z 1].

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- The transformed point is [x' y' z'].

Transformation Matrices in OpenGL

- Transformation matrices in OpenGL are vectors of 16 values (column-major matrices)

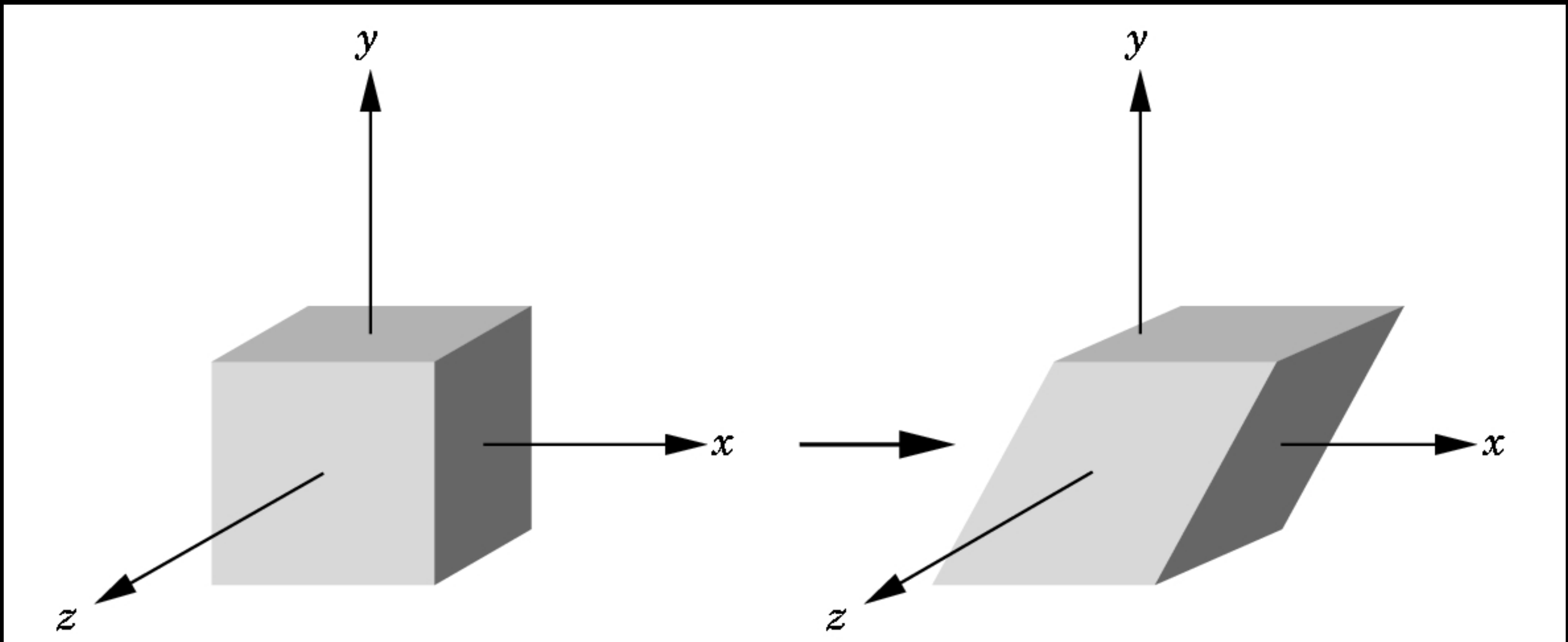
m = {m_1, m_2, ..., m_{16}} represents

$$\begin{bmatrix} m_1 & m_5 & m_9 & m_{13} \\ m_2 & m_6 & m_{10} & m_{14} \\ m_3 & m_7 & m_{11} & m_{15} \\ m_4 & m_8 & m_{12} & m_{16} \end{bmatrix}$$

- Some books transpose all matrices!

Shear Transformations

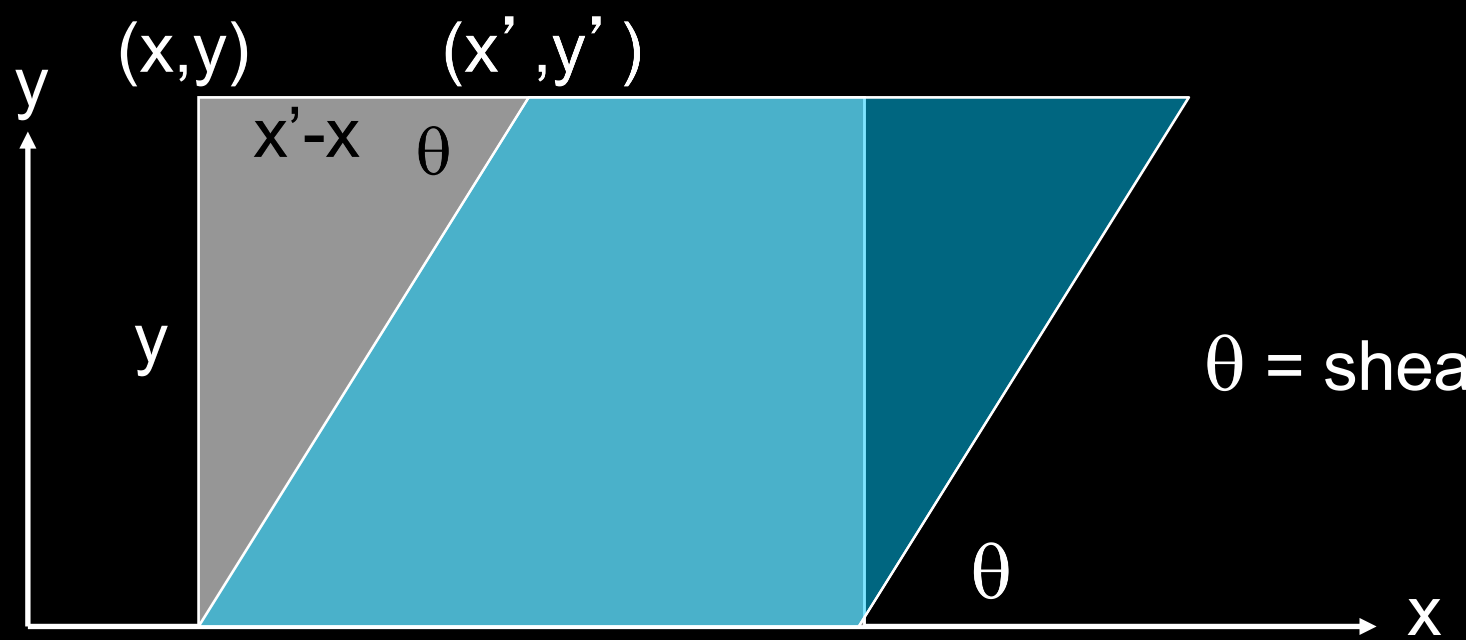
- x-shear scales x proportional to y
- Leaves y and z values fixed



Specification via Shear Angle

- $\cot(\theta) = (x' - x) / y$
- $x' = x + y \cot(\theta)$
- $y' = y$
- $z' = z$

$$H_x(\theta) = \begin{bmatrix} 1 & \cot(\theta) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



θ = shear angle

Specification via Ratios

- For example, shear in both x and z direction
- Leave y fixed
- Slope α for x-shear, γ for z-shear
- Solve

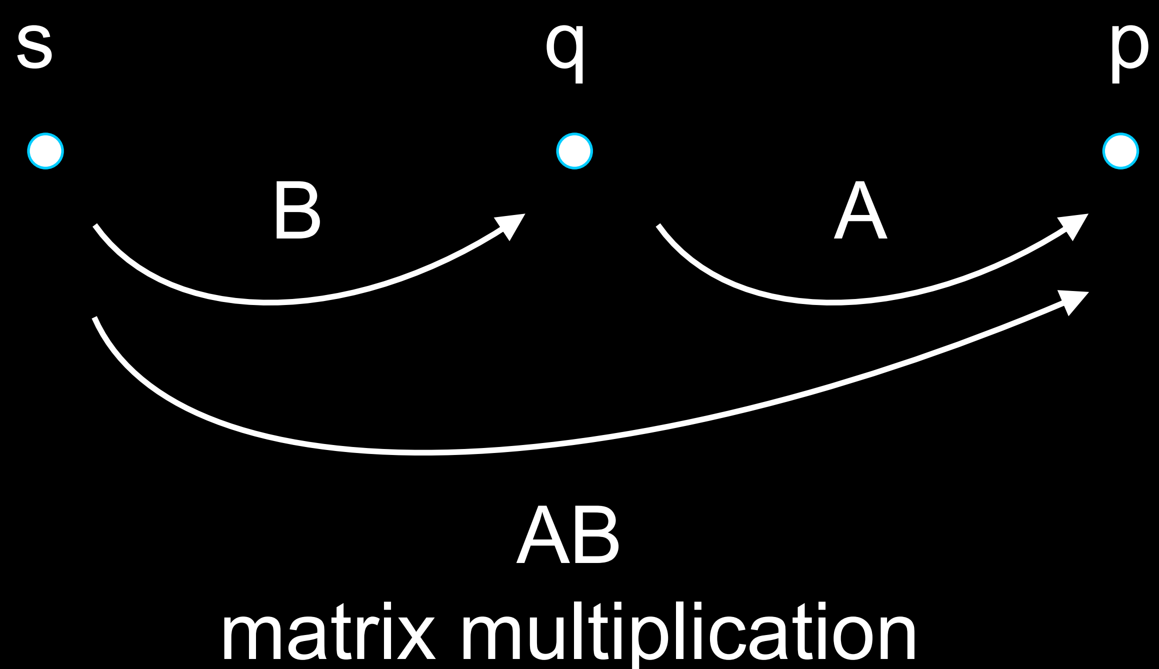
$$H_{x,z}(\alpha,\gamma)\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+\alpha y \\ y \\ z+\gamma y \\ 1 \end{bmatrix}$$

- Yields

$$H_{x,z}(\alpha,\gamma) = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \gamma & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composing Transformations

- Let p = A q, and q = B s.
- Then p = (A B) s .



Composing Transformations

- Fact: Every affine transformation is a composition of rotations, scalings, and translations

- So, how do we compose these to form an x-shear?

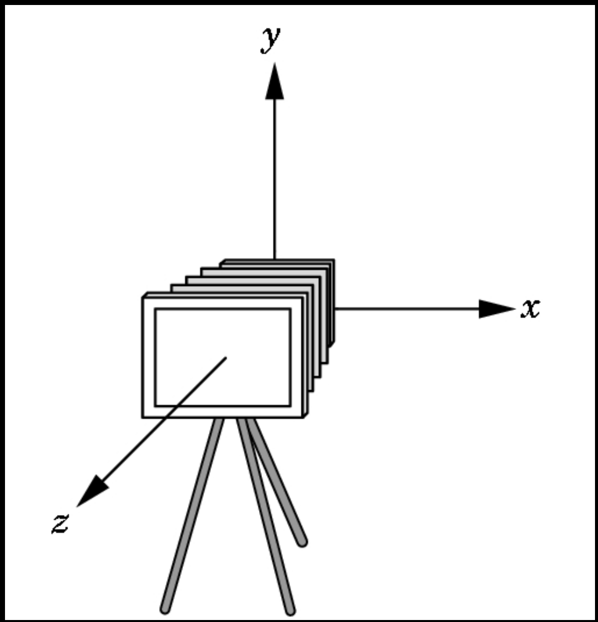
- Exercise!

Outline

- Shear Transformation
- Camera Positioning
- Simple Parallel Projections
- Simple Perspective Projections

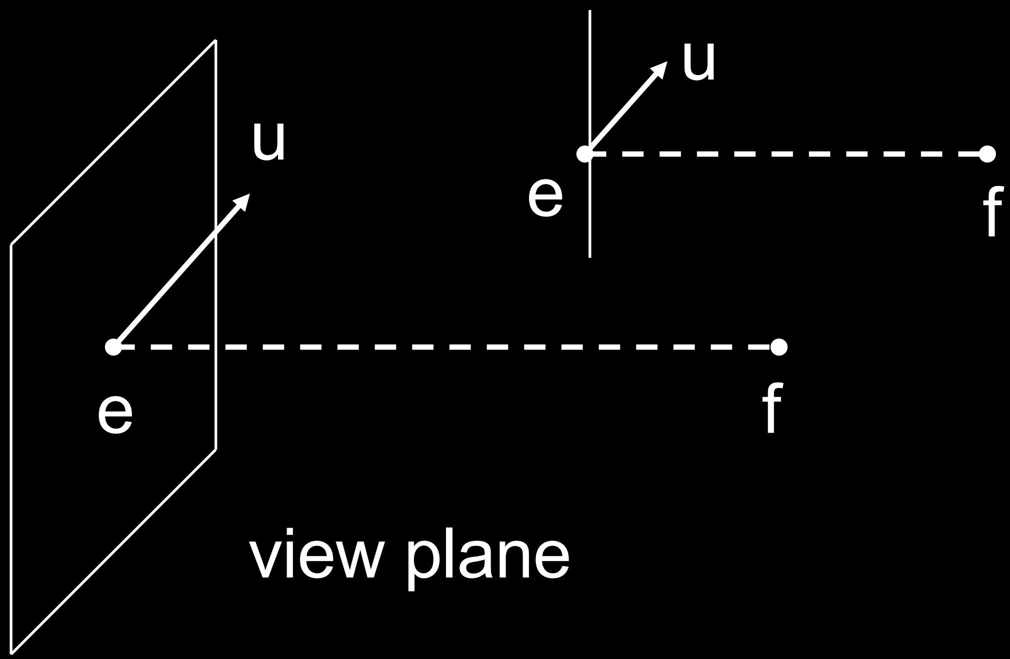
Transform Camera = Transform Scene

- Camera position is identified with a frame
- Either move and rotate the objects
- Or move and rotate the camera
- Initially, camera at origin, pointing in negative z-direction



The Look-At Function

- Convenient way to position camera
- OpenGLMatrix::LookAt(ex, ey, ez,
  fx, fy, fz, ux, uy, uz); // core profile
- gluLookAt(ex, ey, ez,
  fx, fy, fz, ux, uy, uz); // compatibility profile
- e = eye point
- f = focus point
- u = up vector



OpenGL code (camera positioning)

```
void display()
{
  glClear (GL_COLOR_BUFFER_BIT|GL_DEPTH_BUFFER_BIT);
  openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);
  openGLMatrix.LoadIdentity();
  openGLMatrix.LookAt(ex, ey, ez,  fx, fy, fz,  ux, uy, uz);

  openGLMatrix.Translate(x, y, z); // transform the object
  ...
  float m[16]; // column-major
  openGLMatrix.GetMatrix(m); // fill "m" with the matrix entries
  glUniformMatrix4fv(h_modelViewMatrix, 1, GL_FALSE, m);
  renderBunny();
  glutSwapBuffers();
}
```

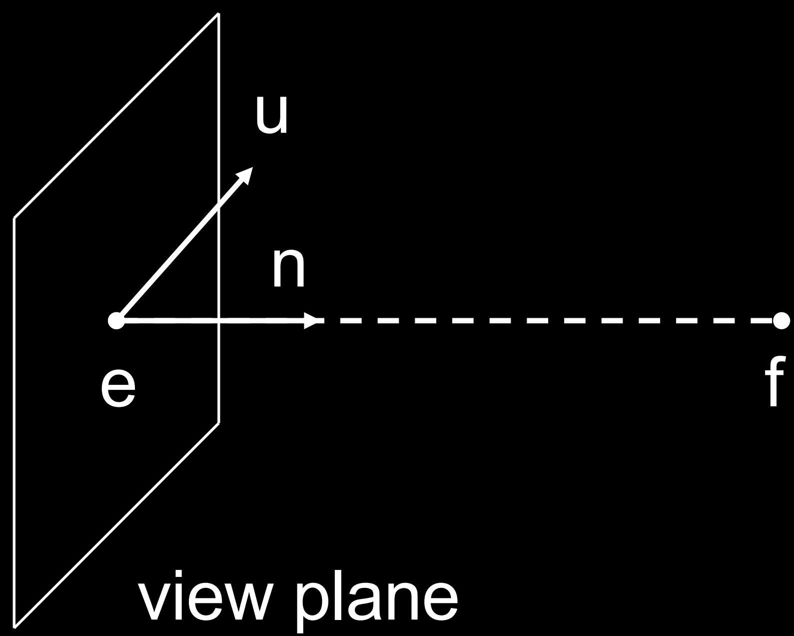
Implementing the Look-At Function

Plan:

1. Transform world frame to camera frame
   - Compose a rotation R with translation T
   - $W = T\,R$
2. Invert W to obtain viewing transformation V
   - $V = W^{-1} = (T\,R)^{-1} = R^{-1}\,T^{-1}$
   - Derive R, then T, then $R^{-1}\,T^{-1}$

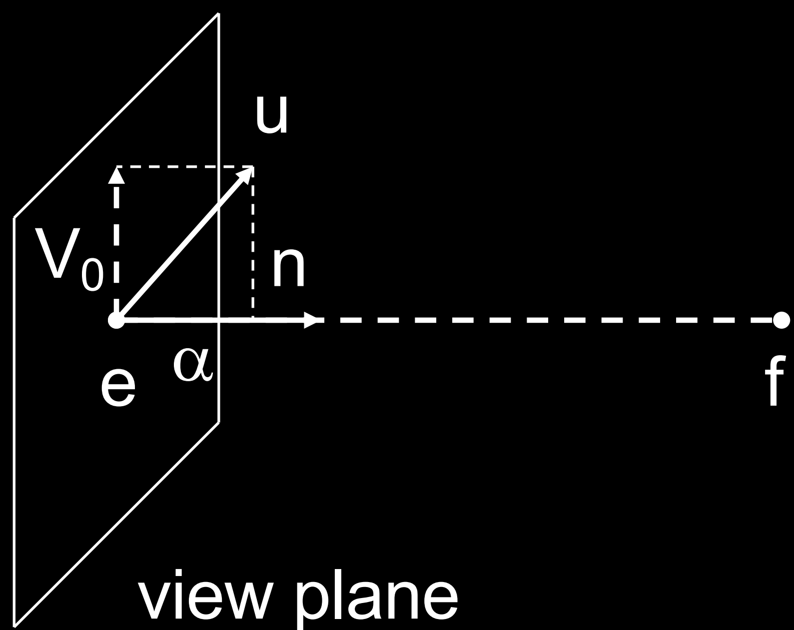
World Frame to Camera Frame I

- Camera points in negative z direction
- n = (f – e) / |f – e| is unit normal to view plane
- Therefore, R maps $[0\ \ 0\ \ -1]^T$ to $[n_x\ \ n_y\ \ n_z]^T$



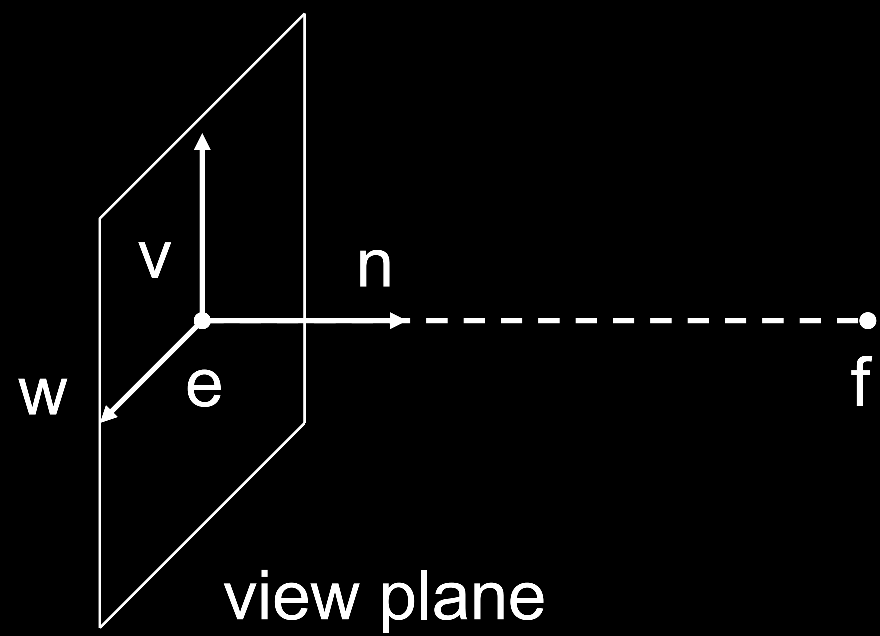
World Frame to Camera Frame II

- R maps $[0,1,0]^T$ to projection of u onto view plane
- This projection v equals:
  - $\alpha = (u \cdot n) / |n| = u \cdot n$
  - $v_0 = u - \alpha\, n$
  - $v = v_0 / |v_0|$



World Frame to Camera Frame III

- Set w to be orthogonal to n and v
- w = n x v
- (w, v, -n) is right-handed



Summary of Rotation

- gluLookAt(e_x, e_y, e_z, f_x, f_y, f_z, u_x, u_y, u_z);
- n = (f – e) / |f – e|
- v = (u – (u · n) n) / |u – (u · n) n|
- w = n x v
- Rotation must map:
  - (1,0,0) to w
  - (0,1,0) to v
  - (0,0,-1) to n

$$\begin{bmatrix} w_x & v_x & -n_x & 0 \\ w_y & v_y & -n_y & 0 \\ w_z & v_z & -n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

World Frame to Camera Frame IV

- Translation of origin to $e = [e_x \;\; e_y \;\; e_z \;\; 1]^T$

$$T = \begin{bmatrix} 1 & 0 & 0 & e_x \\ 0 & 1 & 0 & e_y \\ 0 & 0 & 1 & e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

World Frame to Camera Frame

- $V = W^{-1} = (T\ R)^{-1} = R^{-1}\ T^{-1}$
- R is rotation, so $R^{-1} = R^T$

$$R^{-1} = \begin{bmatrix} w_x & w_y & w_z & 0 \\ v_x & v_y & v_z & 0 \\ -n_x & -n_y & -n_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- T is translation, so T^{-1} negates displacement

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Putting it Together

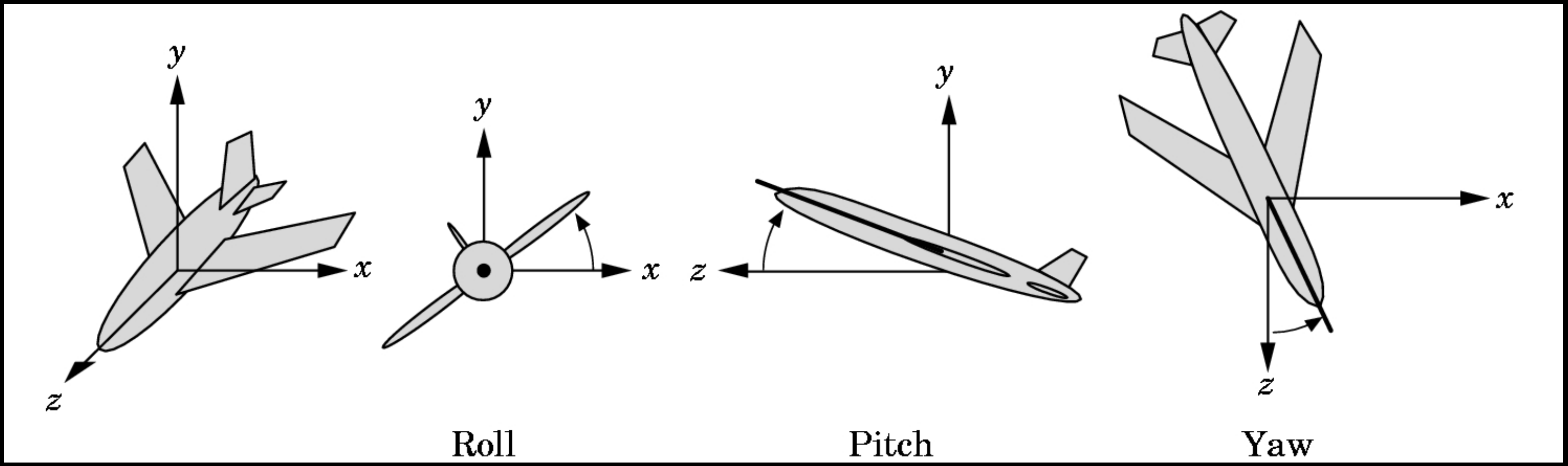
- Calculate $V = R^{-1} T^{-1}$

$$V = \begin{bmatrix} w_x & w_y & w_z & -w_x e_x - w_y e_y - w_z e_z \\ v_x & v_y & v_z & -v_x e_x - v_y e_y - v_z e_z \\ -n_x & -n_y & -n_z & n_x e_x + n_y e_y + n_z e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- This is different from book [Angel, Ch. 5.3.2]
- There, u, v, n are right-handed (here: u, v, -n)

Other Viewing Functions

- Roll (about z), pitch (about x), yaw (about y)



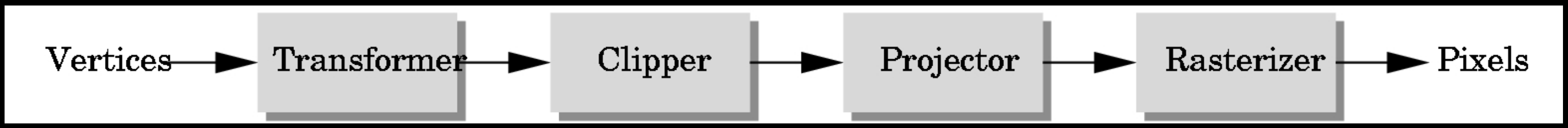
- Assignment 2 poses a related problem

Outline

- Shear Transformation
- Camera Positioning
- Simple Parallel Projections
- Simple Perspective Projections

Projection Matrices

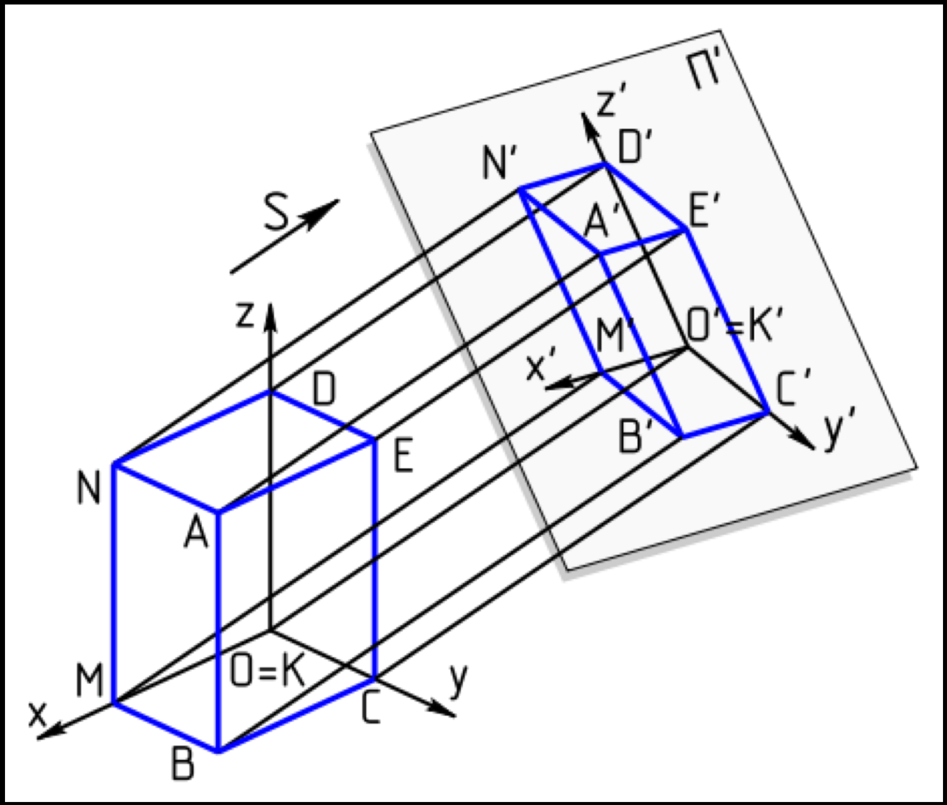
- Recall geometric pipeline



- Projection takes 3D to 2D
- Projections are not invertible
- Projections are described by a 4x4 matrix
- Homogenous coordinates crucial
- Parallel and perspective projections

Parallel Projection

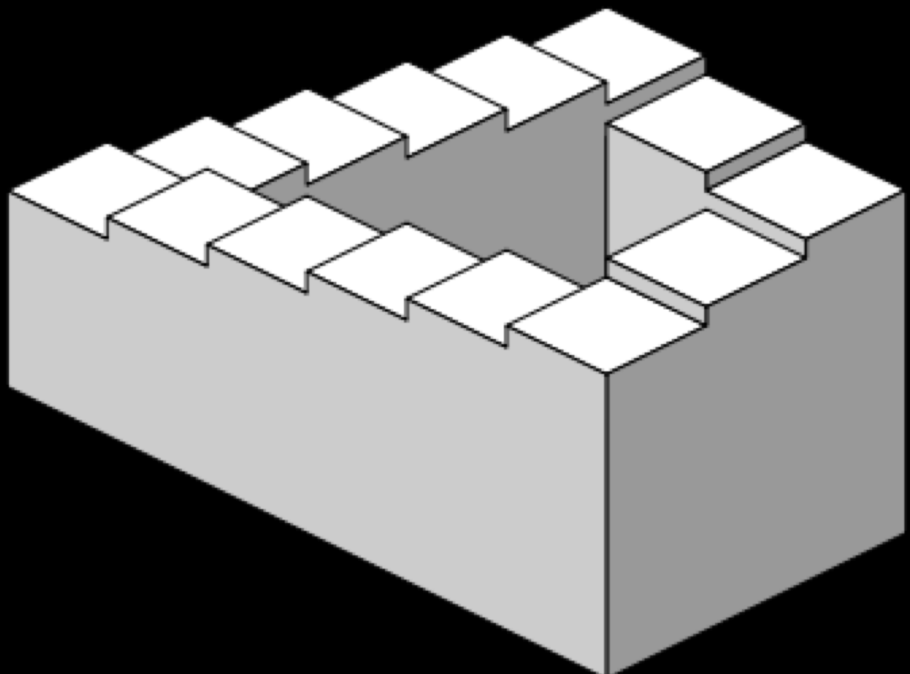
- Project 3D object to 2D via parallel lines
- The lines are not necessarily orthogonal to projection plane



source: Wikipedia

Parallel Projection

- Problem: objects far away do not appear smaller
- Can lead to “impossible objects” :

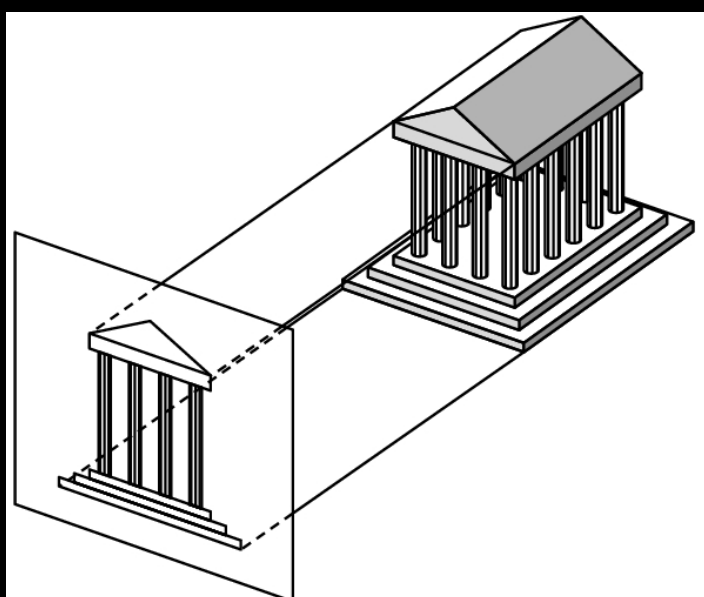


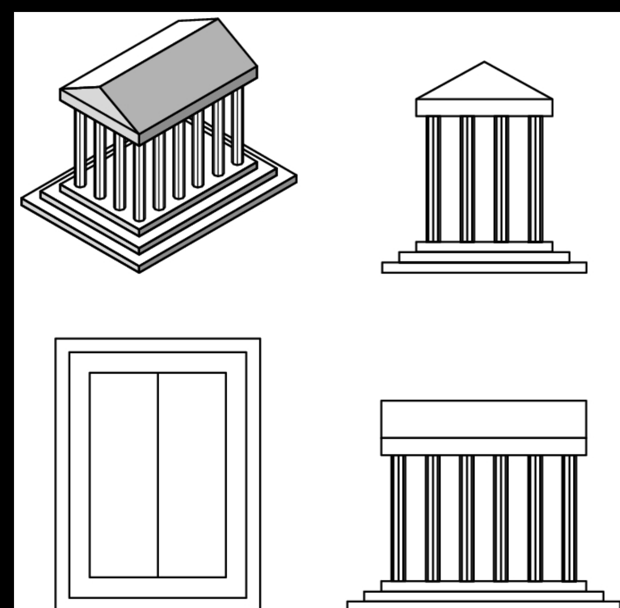
Penrose stairs

source: Wikipedia

Orthographic Projection

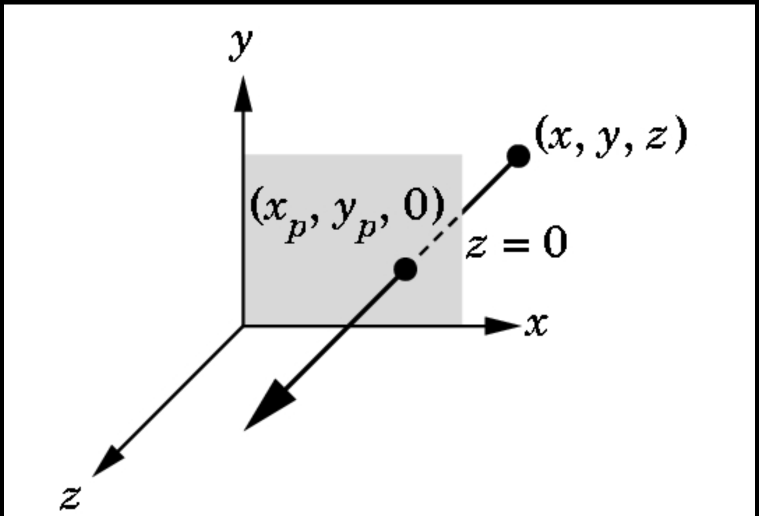
- A special kind of parallel projection: projectors perpendicular to projection plane
- Simple, but not realistic
- Used in blueprints (multiview projections)





Orthographic Projection Matrix

- Project onto z = 0
- $x_p = x$, $y_p = y$, $z_p = 0$
- In homogenous coordinates



$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective

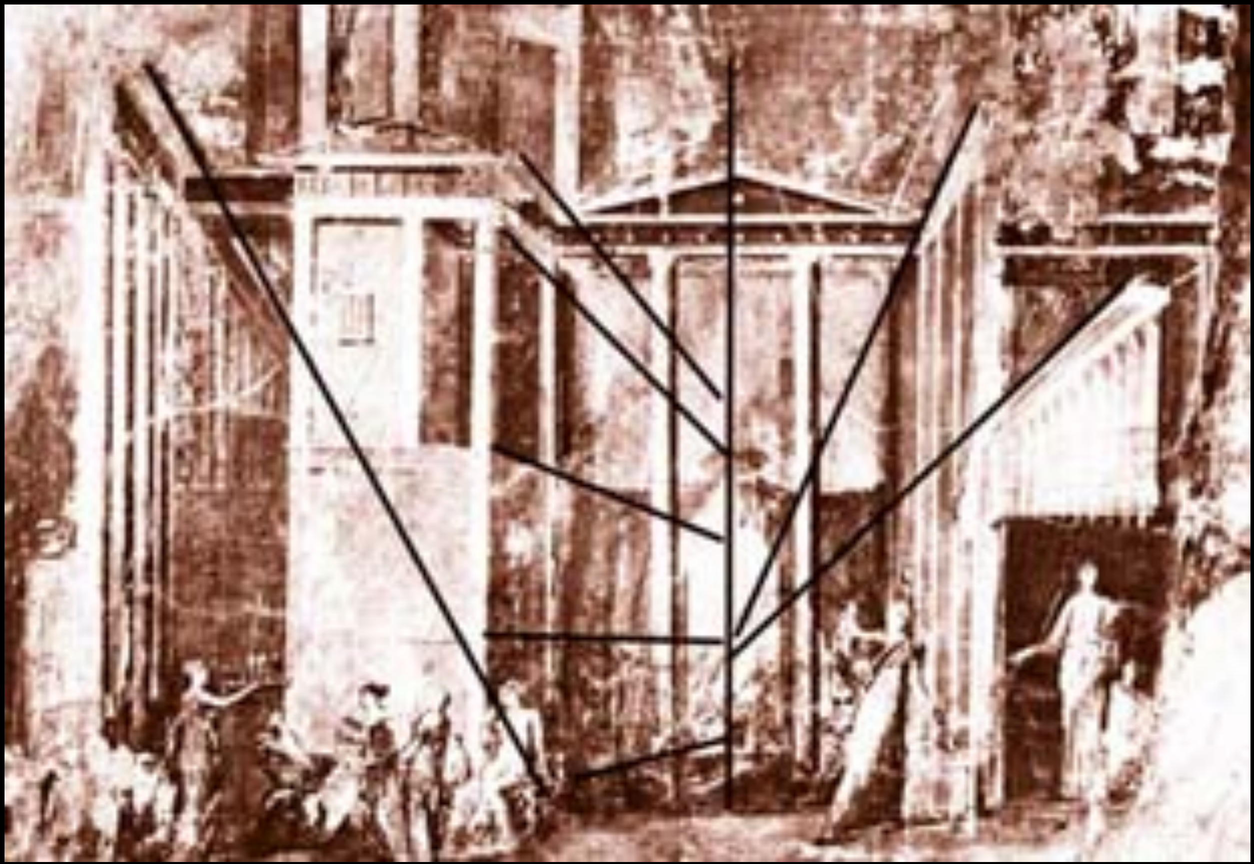
- Perspective characterized by foreshortening
- More distant objects appear smaller
- Parallel lines appear to converge
- Rudimentary perspective in cave drawings:

Lascaux, France

source: Wikipedia

Discovery of Perspective

- Foundation in geometry (Euclid)



Mural from Pompeii, Italy

Middle Ages

- Art in the service of religion
- Perspective abandoned or forgotten

Ottonian manuscript,
ca. 1000

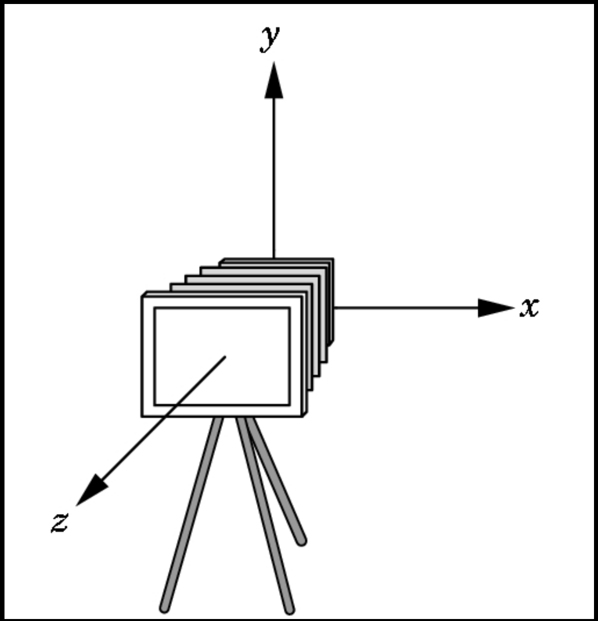
Renaissance

- Rediscovery, systematic study of perspective

Filippo Brunelleschi
Florence, 1415

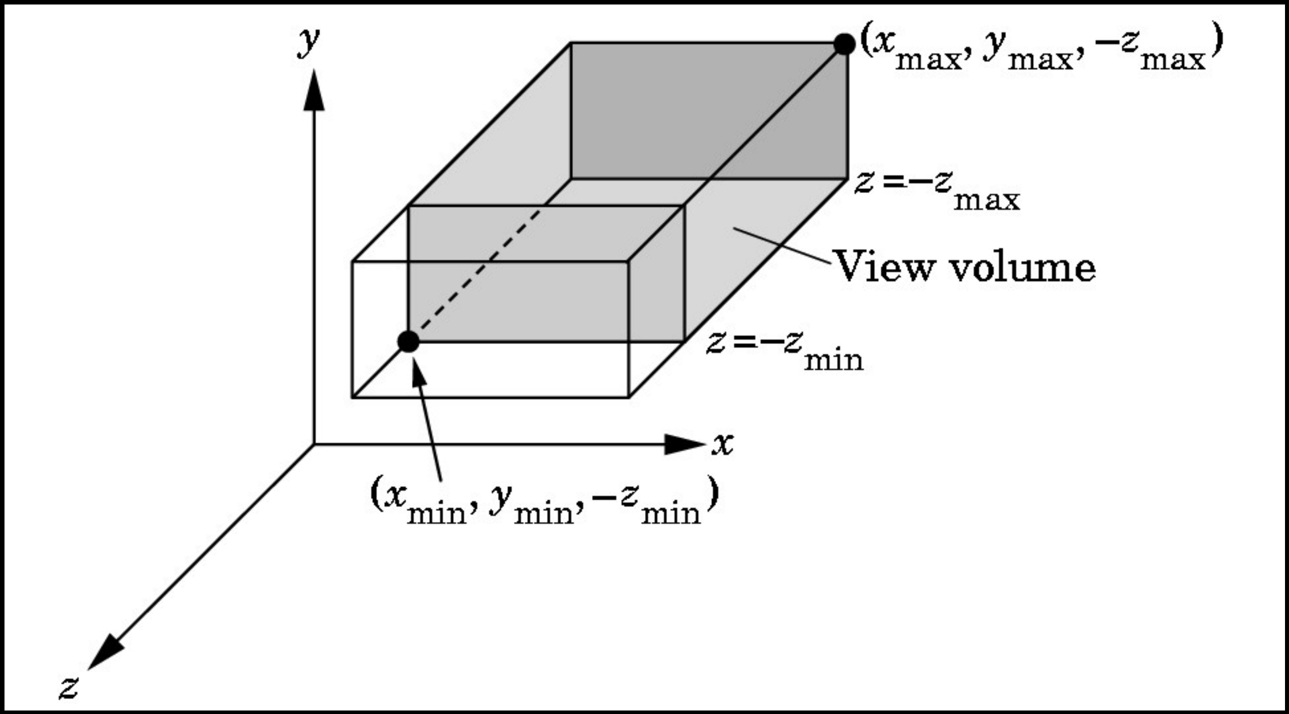
Projection (Viewing) in OpenGL

- Remember: camera is pointing in the negative z direction



Orthographic Viewing in OpenGL (compatibility profile)

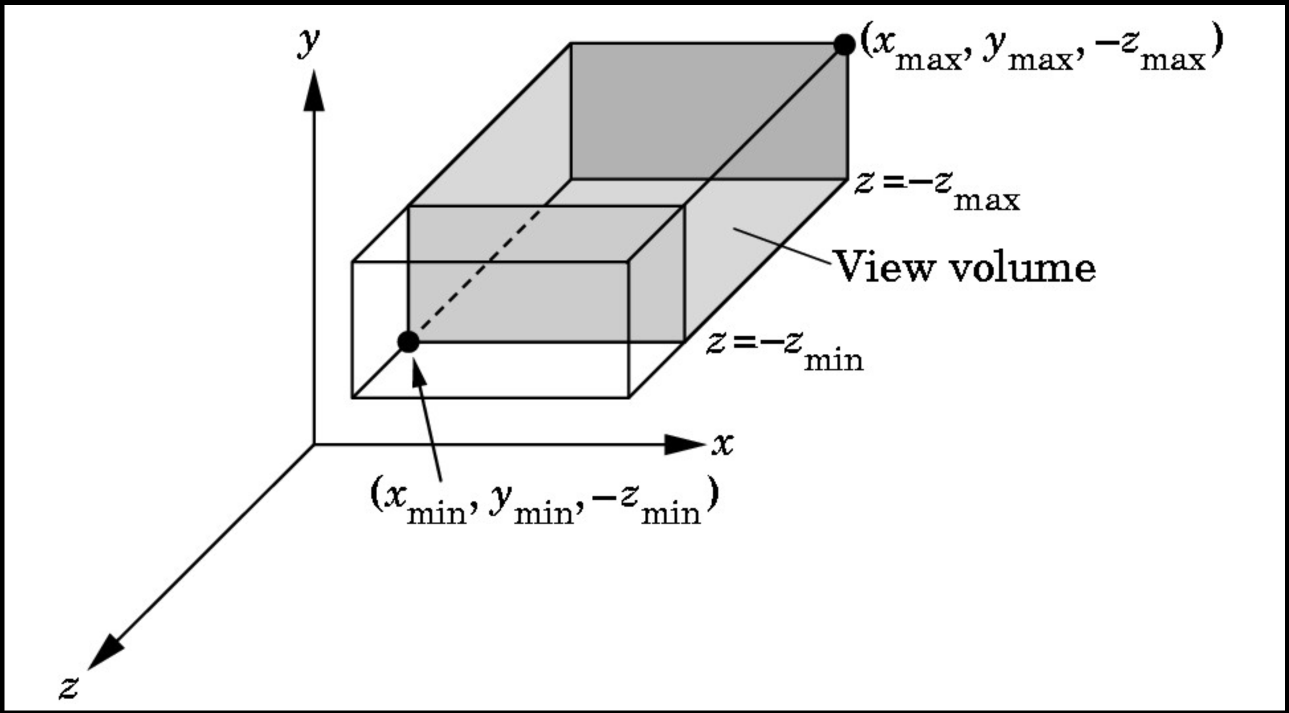
- glOrtho(xmin, xmax, ymin, ymax, near, far)



z_{min} = near, z_{max} = far

Orthographic Viewing in OpenGL (core profile)

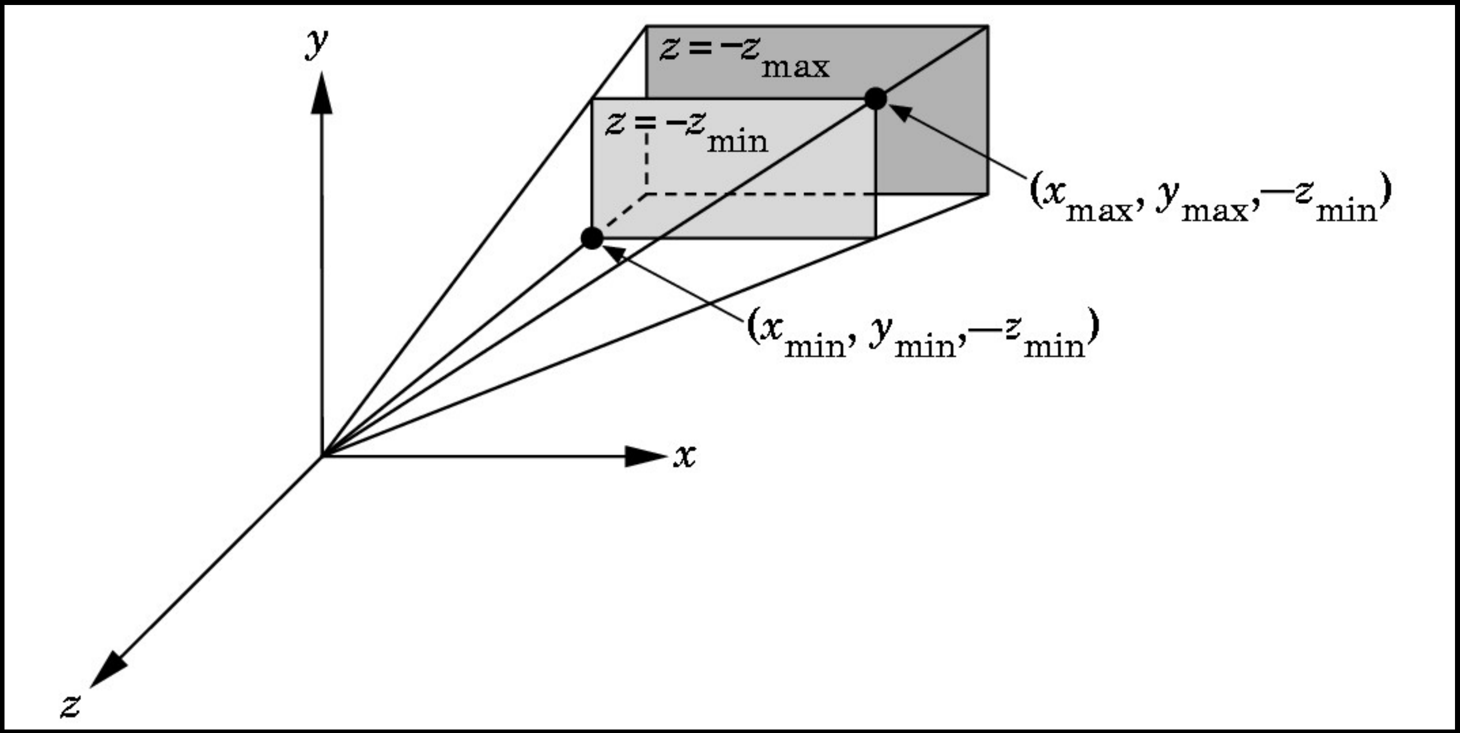
- OpenGLMatrix::Ortho(xmin, xmax, ymin, ymax, near, far)



z_{min} = near, z_{max} = far

Perspective Viewing in OpenGL

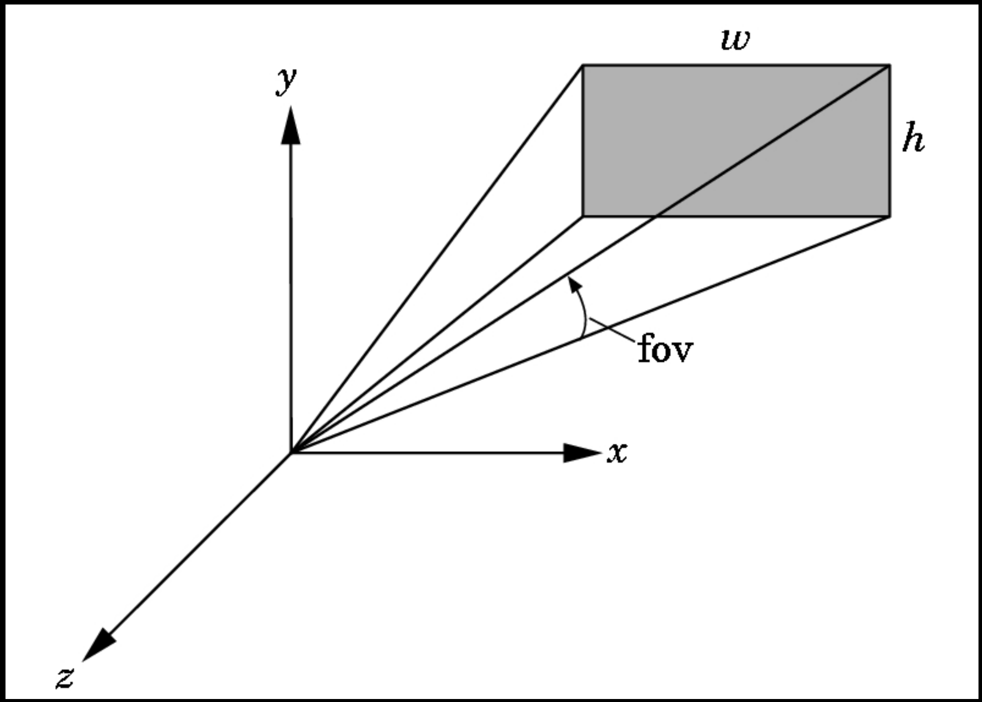
- Two interfaces: glFrustum and gluPerspective
- {OpenGLMatrix::, gl}Frustum(
  xmin, xmax, ymin, ymax, near, far);



z_{min} = near, z_{max} = far

Field of View Interface

- {OpenGLMatrix::, glu}Perspective(
  fovy, aspectRatio, near, far);
- near and far as before
- aspectRatio = w / h
- Fovy specifies field of view as height (y) angle



OpenGL code (reshape)

```
void reshape(int x, int y)
{
  glViewport(0, 0, x, y);

  openGLMatrix.SetMatrixMode(OpenGLMatrix::Projection);
  openGLMatrix.LoadIdentity();
  openGLMatrix.Perspective(60.0, 1.0 * x / y, 0.01, 10.0);
  openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);
}
```

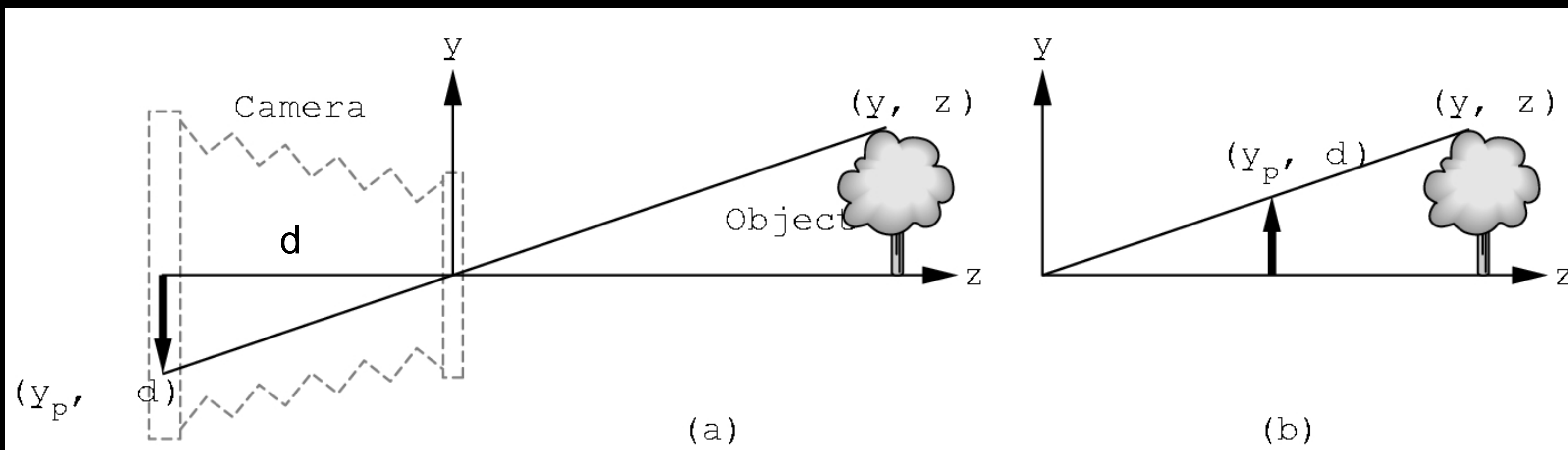
OpenGL code

```
void displayFunc()
{
  …

  openGLMatrix.SetMatrixMode(OpenGLMatrix::Projection);
  float p[16]; // column-major
  openGLMatrix.GetMatrix(p);
  const GLboolean isRowMajor = false;
  glUniformMatrix4fv(h_projectionMatrix, 1, isRowMajor, p);
  …

  renderBunny();
  glutSwapBuffers();
}
```

Perspective Viewing Mathematically



- d = focal length
- $y/z = y_p/d$ so $y_p = y/(z/d) = y\ d\ /\ z$
- Note that y_p is non-linear in the depth z!

Exploiting the 4^{th} Dimension

- Perspective projection is not affine:

$$M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{bmatrix}$$

has no solution for M

- Idea: exploit homogeneous coordinates

$$p = w \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

for arbitrary $w \neq 0$

Perspective Projection Matrix

- Use multiple of point

$$(z/d)\begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix}$$

- Solve

$$M\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \frac{z}{d} \end{bmatrix}$$

with

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} & 0 \end{bmatrix}$$

Projection Algorithm

Input: 3D point (x,y,z) to project

1. Form $[x\ y\ z\ 1]^T$
2. Multiply M with $[x\ y\ z\ 1]^T$; obtaining $[X\ Y\ Z\ W]^T$
3. Perform perspective division:
   X / W, Y / W, Z / W

Output: (X / W, Y / W, Z / W)
(last coordinate will be d)

Perspective Division

- Normalize $[x \; y \; z \; w]^T$ to $[(x/w) \; (y/w) \; (z/w) \; 1]^T$
- Perform perspective division after projection

- Projection in OpenGL is more complex (includes clipping)