

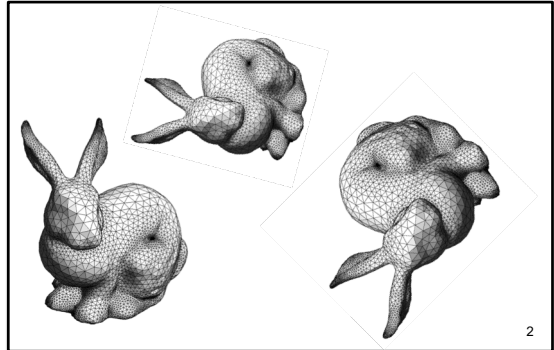
Transformations

Vector Spaces
Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 3]

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University of Southern California

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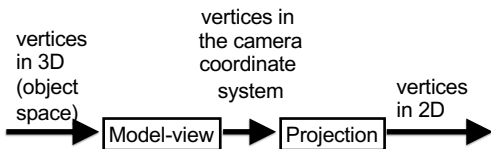
OpenGL Transformations



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OpenGL Transformation Matrices

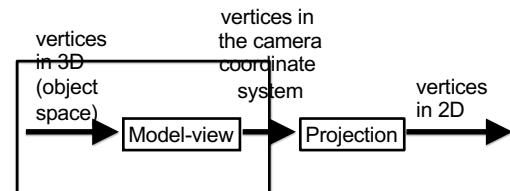
- Model-view matrix (4x4 matrix)
- Projection matrix (4x4 matrix)



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4x4 Model-view Matrix (this lecture)

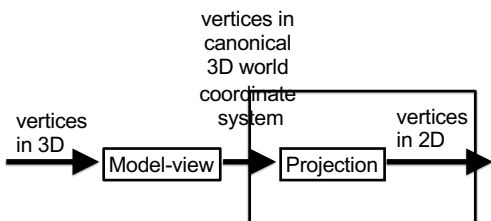
- Translate, rotate, scale objects
- Position the camera



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4x4 Projection Matrix (next lecture)

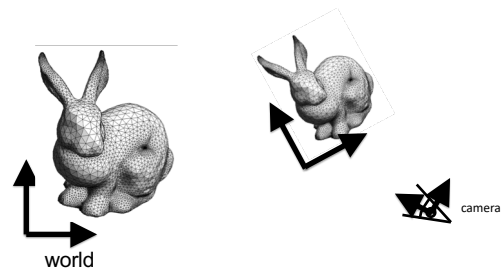
- Project from 3D to 2D



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4x4 Model-view Matrix (this lecture)

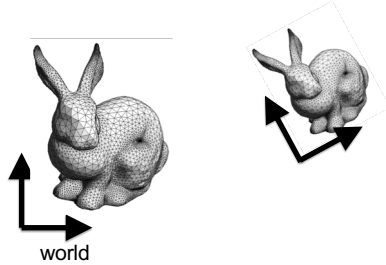
- Translate, rotate, scale objects in world space
- Position and orient the camera



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4x4 Model Matrix

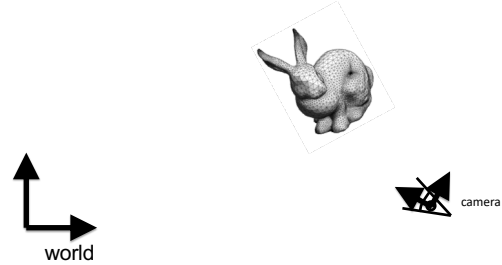
- Translate, rotate, scale objects in world space



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4x4 View Matrix

- Position and orient the camera
- From world space to camera space



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OpenGL Transformation Matrices



- Manipulated separately in OpenGL
- Core profile: set them directly
- Compatibility profile: must set matrix mode

```
glMatrixMode (GL_MODELVIEW);  
glMatrixMode (GL_PROJECTION);
```

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Setting the Modelview Matrix: Core Profile

- Set identity:

```
openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);  
openGLMatrix.LoadIdentity();
```
- Use our openGLMatrix library functions:

```
openGLMatrix.Translate(dx, dy, dz);  
openGLMatrix.Rotate(angle, vx, vy, vz);  
openGLMatrix.Scale(sx, sy, sz);
```
- Upload m to the GPU:

```
float m[16]; // column-major  
openGLMatrix.GetMatrix(m);  
GLboolean isRowMajor = GL_FALSE;  
glUniformMatrix4fv(h_modelViewMatrix, 1, isRowMajor, m);  
// note: h_modelViewMatrix is a handle of the shader  
modelview matrix variable (will discuss in Shaders lecture)
```

Setting the Modelview Matrix: Compatibility Profile

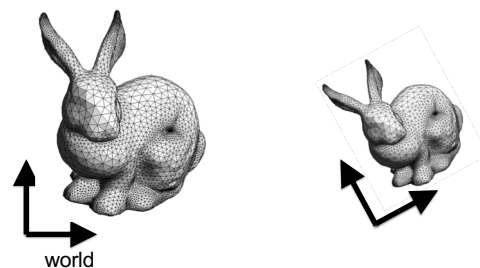
- Load or post-multiply

```
glMatrixMode (GL_MODELVIEW);  
glLoadIdentity(); // very common usage  
float m[16] = { ... };  
glLoadMatrixf(m); // rare, advanced  
glMultMatrixf(m); // rare, advanced
```
- Use library functions

```
glTranslatef(dx, dy, dz);  
glRotatef(angle, vx, vy, vz);  
glScalef(sx, sy, sz);
```

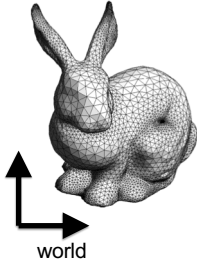
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Translated, rotated, scaled object



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The rendering coordinate system

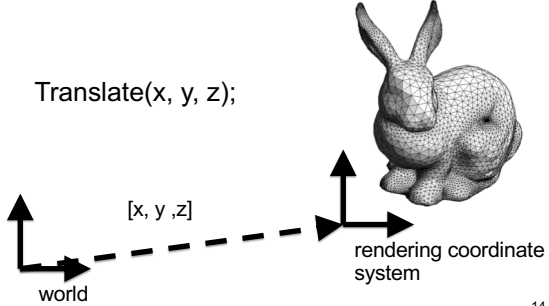


Initially (after LoadIdentity()) :
rendering coordinate system = world coordinate system

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The rendering coordinate system

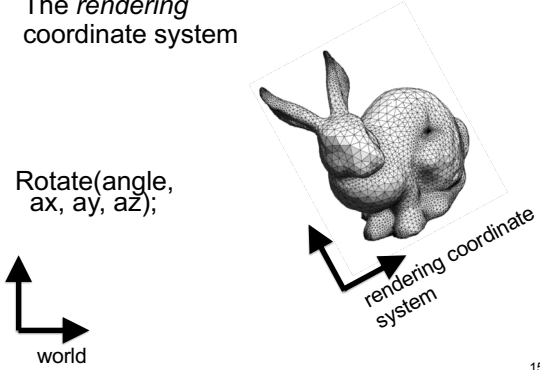
Translate(x, y, z);



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The rendering coordinate system

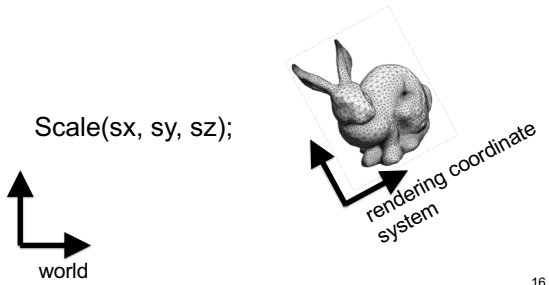
Rotate(angle, ax, ay, az);



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The rendering coordinate system

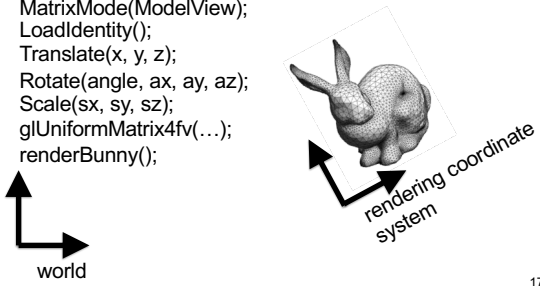
Scale(sx, sy, sz);



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OpenGL pseudo-code

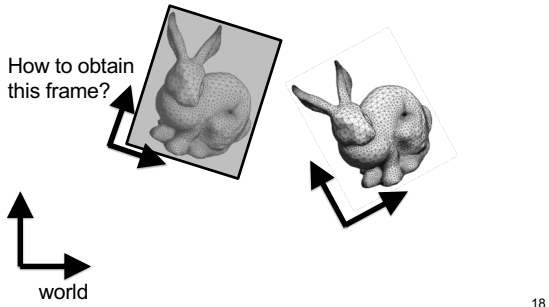
```
MatrixMode(ModelView);
LoadIdentity();
Translate(x, y, z);
Rotate(angle, ax, ay, az);
Scale(sx, sy, sz);
glUniformMatrix4fv(...);
renderBunny();
```



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Rendering more objects

How to obtain this frame?



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Rendering more objects

Solution 1:
Find Translate(...), Rotate(...),
Scale(...)

How to obtain this frame?

world

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Rendering more objects

Solution 2:
LoadIdentity();
Find Translate(...), Rotate(...), Scale(...)

How to obtain this frame?

world

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3D Math Review

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Scalars

- Scalars α, β, γ from a *scalar field*
- Operations $\alpha + \beta, \alpha \cdot \beta, 0, 1, -\alpha, ()^{-1}$
- "Expected" laws apply
- Examples: rationals or reals with addition and multiplication

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Vectors

- Vectors u, v, w from a *vector space*
- Vector addition $u + v$, subtraction $u - v$
- Zero vector $\mathbf{0}$
- Scalar multiplication αv

(a) (b)

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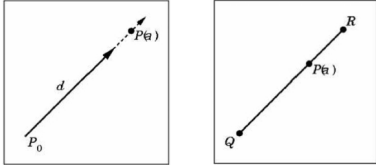
Euclidean Space

- Vector space over real numbers
- Three-dimensional in computer graphics
- Dot product: $\alpha = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- u, v are *orthogonal* if $u \cdot v = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of v

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Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha d$

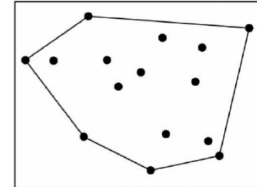


- Line segment between Q and R :
 $P(\alpha) = (1-\alpha)Q + \alpha R$ for $0 \leq \alpha \leq 1$

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Convex Hull

- Convex hull defined by
 $P = \alpha_1 P_1 + \dots + \alpha_n P_n$
for $\alpha_1 + \dots + \alpha_n = 1$
and $0 \leq \alpha_i \leq 1, i = 1, \dots, n$



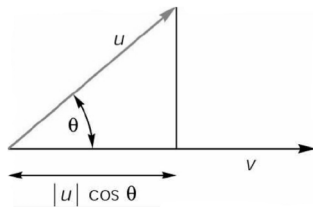
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Projection

- Dot product projects one vector onto another vector

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u| |v| \cos(\theta)$$

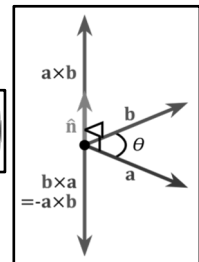
$$pr_v u = (u \cdot v) v / |v|^2$$



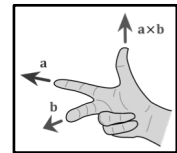
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Cross Product

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$



- $|a \times b| = |a| |b| |\sin(\theta)|$
- Cross product is perpendicular to both a and b
- Right-hand rule



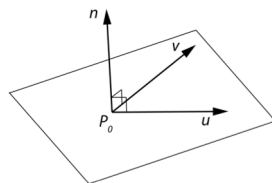
Source: Wikipedia

Plane

- Plane defined by point P_0 and vectors u and v

- u and v should not be parallel

- Parametric form:
 $T(\alpha, \beta) = P_0 + \alpha u + \beta v$
(α and β are scalars)



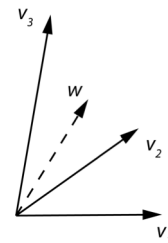
- $n = u \times v / |u \times v|$ is the normal
- $n \cdot (P - P_0) = 0$ if and only if P lies in plane

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Coordinate Systems

- Let v_1, v_2, v_3 be three linearly independent vectors in a 3-dimensional vector space

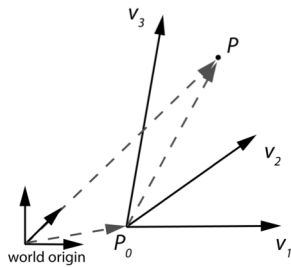
- Can write any vector w as
 $w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
for some scalars $\alpha_1, \alpha_2, \alpha_3$



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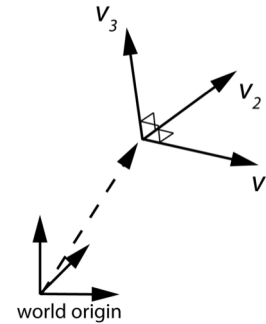
Frames

- Frame = origin P_0 + coordinate system
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$



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In Practice, Frames are Often Orthogonal



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Representing 3D transformations (and model-view matrices)

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Linear Transformations

- 3 x 3 matrices represent linear transformations
 $\mathbf{a} = \mathbf{M}\mathbf{b}$
- Can represent rotation, scaling, and reflection
- Cannot represent translation

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

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In order to represent rotations, scales AND translations: Homogeneous Coordinates

- Augment $[\alpha_1 \ \alpha_2 \ \alpha_3]^T$ by adding a fourth component (1):

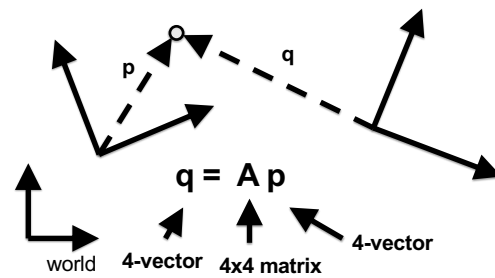
$$\mathbf{p} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 1]^T$$

- Homogeneous property:

$$\mathbf{p} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 1]^T = [\beta\alpha_1 \ \beta\alpha_2 \ \beta\alpha_3 \ \beta]^T, \text{ for any scalar } \beta \neq 0$$

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Homogeneous coordinates are transformed by 4x4 matrices



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Affine Transformations (4x4 matrices)

- Translation
- Rotation
- Scaling
- Any composition of the above
- Later: projective (perspective) transformations
 - Also expressible as 4 x 4 matrices!

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Translation

- $\mathbf{q} = \mathbf{p} + \mathbf{d}$ where $\mathbf{d} = [\alpha_x \ \alpha_y \ \alpha_z \ 0]^T$
- $\mathbf{p} = [x \ y \ z \ 1]^T$
- $\mathbf{q} = [x' \ y' \ z' \ 1]^T$
- Express in matrix form $\mathbf{q} = \mathbf{T} \mathbf{p}$ and solve for \mathbf{T}

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Scaling

- $x' = \beta_x x$
- $y' = \beta_y y$
- $z' = \beta_z z$
- Express as $\mathbf{q} = \mathbf{S} \mathbf{p}$ and solve for \mathbf{S}

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Rotation in 2 Dimensions

- Rotation by θ about the origin
- $x' = x \cos \theta - y \sin \theta$
- $y' = x \sin \theta + y \cos \theta$
- Express in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note that the determinant is 1

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Rotation in 3 Dimensions

- Orthogonal matrices:

$$RR^T = R^T R = I \\ \det(R) = 1$$

- Affine transformation:

$$A = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Affine Matrices are Composed by Matrix Multiplication

- $\mathbf{A} = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3$
- Applied from right to left
- $\mathbf{A} \mathbf{p} = (\mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3) \mathbf{p} = \mathbf{A}_1 (\mathbf{A}_2 (\mathbf{A}_3 \mathbf{p}))$
- Compatibility mode:
When calling `glTranslate3f`, `glRotatef`, or `glScalef`, OpenGL forms the corresponding 4x4 matrix, and multiplies the current modelview matrix with it.

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Summary

- OpenGL Transformation Matrices
- Vector Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices

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