CSCI 420 Computer Graphics
Lecture 10

Splines

Hermite Splines
Bezier Splines
Catmull-Rom Splines
Other Cubic Splines
[Angel Ch. 10]

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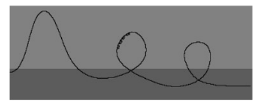
Roller coaster

- Next programming assignment involves creating a 3D roller coaster animation
- We must model the 3D curve describing the roller coaster, but how?



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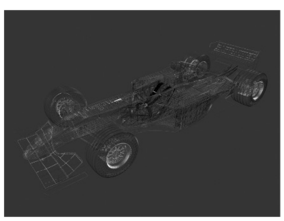
Modeling Complex Shapes

- We want to build models of very complicated objects
- Complexity is achieved using simple pieces
  - polygons,
  - parametric curves and surfaces, or
  - implicit curves and surfaces
- This lecture: **parametric curves**



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What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

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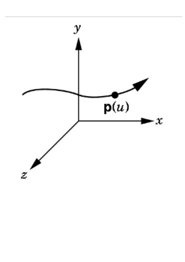
Curve Representations

- Explicit: y = f(x) $y = x^2$ $y = mx + b$
  - Must be a function (single-valued)
  - Big limitation—vertical lines?
- Parametric: (x,y,z) = (f(u),g(u),h(u))
  - \+ Easy to specify, modify, control
  - Extra "hidden" variable u, the *parameter*

  $$(x, y) = (\cos u, \sin u)$$
- Implicit (2D): f(x,y) = 0
  - \+ y can be a multiple valued function of x
  - Hard to specify, modify, control

  $$x^2 + y^2 - r^2 = 0$$



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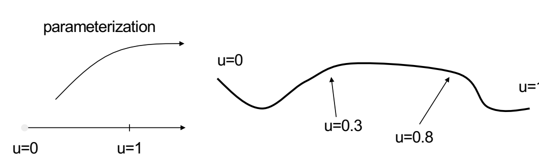
Parameterization of a Curve

- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.
- Parameterization is not unique. It can be slow, fast, with continuous / discontinuous speed, clockwise (CW) or CCW…



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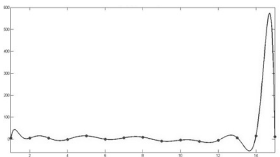
Polynomial Interpolation

- An *n*-th degree polynomial fits a curve to n+1 points
  - called Lagrange Interpolation
  - result is a curve that is too wiggly, change to any control point affects entire curve (non-local)
  - *this method is poor*
- We usually want the curve to be as smooth as possible
  - minimize the wiggles
  - high-degree polynomials are bad



source: Wikipedia

Lagrange interpolation, degree=14

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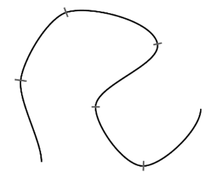
Splines: Piecewise Polynomials

- A spline is a *piecewise polynomial:* Curve is broken into consecutive segments, each of which is a low-degree polynomial interpolating (passing through) the control points
- *Cubic* piecewise polynomials are the most common:
  - They are the lowest order polynomials that
    1. interpolate two points and
    2. allow the gradient at each point to be defined (C^1 continuity is possible).
  - Piecewise definition gives local control.
  - Higher or lower degrees are possible, of course.



a spline

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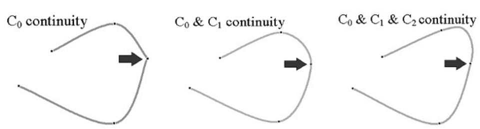
Piecewise Polynomials

- Spline: many polynomials pieced together
- Want to make sure they fit together nicely



Continuous in position

Continuous in position and tangent vector

Continuous in position, tangent, and curvature

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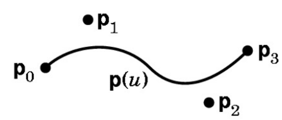
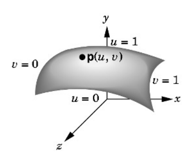
Splines

- Types of splines:
  - Hermite Splines
  - Bezier Splines
  - Catmull-Rom Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS
- Splines can be used to model both curves and surfaces





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Cubic Curves in 3D

- Cubic polynomial:
  - $p(u) = au^3+bu^2+cu+d = [u^3\ \ u^2\ \ u\ \ 1]\ [a\ \ b\ \ c\ \ d]^T$
  - a,b,c,d are 3-vectors, u is a scalar
- Three cubic polynomials, one for each coordinate:
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- In matrix notation:

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

- Or simply: $p = [u^3\ u^2\ u\ 1]\ A$

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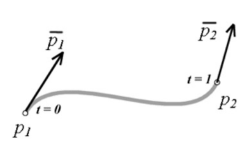
Cubic Hermite Splines



Hermite Specification

We want a way to specify the end points and the slope at the end points!

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Deriving Hermite Splines

- Four constraints: value and slope (in 3-D, position and tangent vector) at beginning and end of interval [0,1] :
  - $p(0) = p_1 = (x_1, y_1, z_1)$
  - $p(1) = p_2 = (x_2, y_2, z_2)$
  - $p'(0) = \overline{p}_1 = (\overline{x}_1, \overline{y}_1, \overline{z}_1)$
  - $p'(1) = \overline{p}_2 = (\overline{x}_2, \overline{y}_2, \overline{z}_2)$

  ← the user constraints
- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$
- Four unknowns: *a, b, c, d*

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Deriving Hermite Splines

- Assume cubic form: $p(u) = au^3 + bu^2 + cu + d$

  $p_1 = p(0) = d$

  $p_2 = p(1) = a + b + c + d$

  $\overline{p}_1 = p'(0) = c$

  $\overline{p}_2 = p'(1) = 3a + 2b + c$
- Linear system: 12 equations for 12 unknowns (however, can be simplified to 4 equations for 4 unknowns)
- Unknowns: *a, b, c, d* (each of *a*, *b*, *c*, *d* is a 3-vector)

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Deriving Hermite Splines

$$\begin{aligned} d &= p_1 \\ a + b + c + d &= p_2 \\ c &= \overline{p}_1 \\ 3a + 2b + c &= \overline{p}_2 \end{aligned}$$

Rewrite this 12x12 system as a 4x4 system:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \overline{x}_1 & \overline{y}_1 & \overline{z}_1 \\ \overline{x}_2 & \overline{y}_2 & \overline{z}_2 \end{bmatrix}$$

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The Cubic Hermite Spline Equation

- After inverting the 4x4 matrix, we obtain:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \overline{x}_1 & \overline{y}_1 & \overline{z}_1 \\ \overline{x}_2 & \overline{y}_2 & \overline{z}_2 \end{bmatrix}$$

point on the spline — parameter vector — basis — control matrix (what the user gets to pick)

- This form is typical for splines
  - basis matrix and meaning of control matrix change with the spline type

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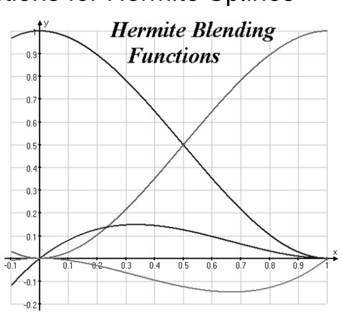
Four Basis Functions for Hermite Splines

transpose

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \overline{p}_1 \\ \overline{p}_2 \end{bmatrix}$$

4 Basis Functions



Every cubic Hermite spline is a linear combination (blend) of these 4 functions.

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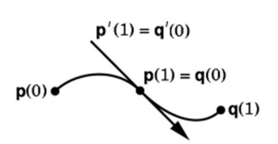
Piecing together Hermite Splines

It's easy to make a multi-segment Hermite spline:

- each segment is specified by a cubic Hermite curve
- just specify the position and tangent at each "joint" (called *knot*)
- the pieces fit together with matched positions and first derivatives
- gives C1 continuity



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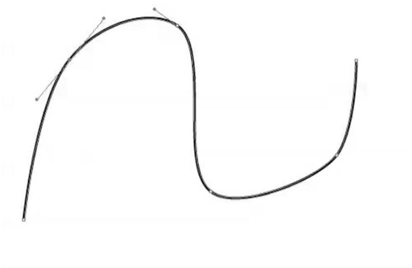
Hermite Splines in Adobe Illustrator



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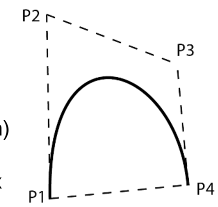
Bezier Splines

- Variant of the Hermite spline
- Instead of endpoints and tangents, four control points
  - points P1 and P4 are on the curve
  - points P2 and P3 are off the curve
  - p(0) = P1, p(1) = P4,
  - p'(0) = 3(P2-P1), p'(1) = 3(P4 - P3)
- Basis matrix is derived from the Hermite basis (or from scratch)
- Convex Hull property: curve contained within the convex hull of control points
- Scale factor "3" is chosen to make "velocity" approximately constant



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The Bezier Spline Matrix

$$[x\ y\ z] = [u^3\ u^2\ u\ 1]\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

Hermite basis — Bezier to Hermite — Bezier control matrix

$$= [u^3\ u^2\ u\ 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

Bezier basis — Bezier control matrix

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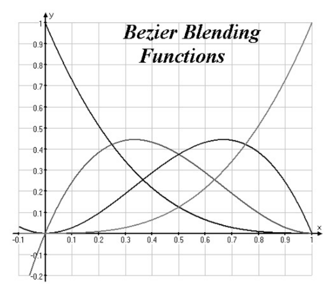
Bezier Blending Functions



$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{T}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

Also known as the order 4, degree 3 Bernstein polynomials

Nonnegative, sum to 1

The entire curve lies inside the polyhedron bounded by the control points

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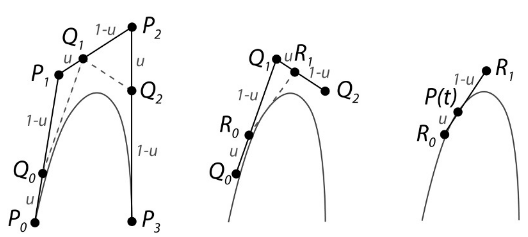
DeCasteljau Construction



Efficient algorithm to evaluate Bezier splines.
Similar to Horner rule for polynomials.
Can be extended to interpolations of 3D rotations.

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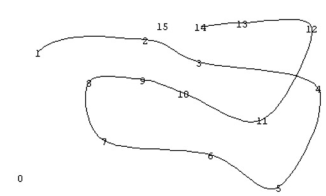
Catmull-Rom Splines

- Roller-coaster (next programming assignment)
- With Hermite splines, the designer must arrange for consecutive tangents to be collinear, to get C^1 continuity. Similar for Bezier. This gets tedious.
- Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.
- Compared to Hermite/Bezier: fewer control points required, but less freedom.



Catmull-Rom spline

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Constructing the Catmull-Rom Spline

Suppose we are given n control points in 3-D: $p_1, p_2, \ldots, p_n$.

For a Catmull-Rom spline, we set the tangent at p_i to $s * (p_{i+1} - p_{i-1})$ for i=2, ..., n-1, for some s (often s=0.5)

s is *tension parameter*: determines the magnitude (but not direction!) of the tangent vector at point p_i

What about endpoint tangents? Use extra control points p_0, p_{n+1} .

Now we have positions and tangents at each knot. This is a Hermite specification. Now, just use Hermite formulas to derive the spline.

Note: curve between p_i and p_{i+1} is completely determined by p_{i-1}, p_i, p_{i+1}, p_{i+2} .

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Catmull-Rom Spline Matrix

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control matrix}}$$

- Derived in way similar to Hermite and Bezier
- Parameter s is typically set to s=1/2.

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Splines with More Continuity?

- So far, only C^1 continuity.
- How could we get C^2 continuity at control points?

- Possible answers:
  - Use higher degree polynomials
    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - Give up local control → natural cubic splines
    A change to any control point affects the entire curve
  - Give up interpolation → cubic B-splines
    Curve goes near, but not through, the control points

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Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

Summary:

Cannot get C2, interpolation and local control with cubics

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Natural Cubic Splines

- If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines*

- It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)

- Finding all the right weights is a *global* calculation (solve tridiagonal linear system)

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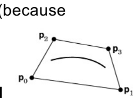
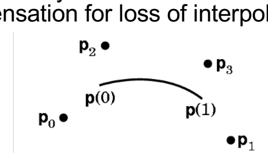
B-Splines

- Give up interpolation
  - the curve passes *near* the control points
  - best generated with interactive placement (because it's hard to guess where the curve will go)
- Curve obeys the convex hull property
- C2 continuity and local control are good compensation for loss of interpolation





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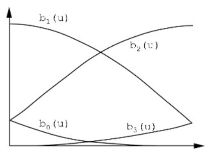
B-Spline Basis

- We always need 3 more control points than the number of spline segments

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bsi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



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Other Common Types of Splines

- Non-uniform Splines
- Non-Uniform Rational B-Splines (NURBS)
- NURBS are very popular and used in many commercial packages

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How to Draw Spline Curves

- Basis matrix equation allows same code to draw any spline type
- Method 1: brute force
  - Calculate the coefficients
  - For each cubic segment, vary *u* from *0* to *1* (fixed step size)
  - Plug in *u* value, matrix multiply to compute position on curve
  - Draw line segment from last position to current position
- What's wrong with this approach?
  - Draws in even steps of u
  - Even steps of u does not mean even steps of x
  - Line length will vary over the curve
  - Want to bound line length
    - » too long: curve looks jagged
    - » too short: curve is slow to draw

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Drawing Splines, 2

- Method 2: recursive subdivision - vary step size to draw short lines

```
Subdivide(u0,u1,maxlinelength)
   umid = (u0 + u1)/2
   x0 = F(u0)
   x1 = F(u1)
   if |x1 - x0| > maxlinelength
      Subdivide(u0,umid,maxlinelength)
      Subdivide(umid,u1,maxlinelength)
   else drawline(x0,x1)
```

- Variant on Method 2 - subdivide based on curvature
  - replace condition in "if" statement with straightness criterion
  - draws fewer lines in flatter regions of the curve

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Summary

- Piecewise cubic is generally sufficient
- Define conditions on the curves and their continuity
- Most important:
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines p(u) = u B G
  - given a definition, derive a basis matrix (do not memorize the matrices themselves)

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