INVERSE KINEMATICS AND GEOMETRIC CONSTRAINTS FOR ARTICULATED FIGURE MANIPULATION

by

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Abstract

Computer animation of articulated figures can be tedious, largely due to the amount of data which must be specified at each frame. Animation techniques range from simple interpolation between keyframed figure poses to higher-level algorithmic models of specific movement patterns. The former provides the animator with complete control over the movement, whereas the latter may provide only limited control via some high-level parameters incorporated into the model. Inverse kinematic techniques adopted from the robotics literature have the potential to relieve the animator of detailed specification of every motion parameter within a figure, while retaining complete control over the movement, if desired.

This work investigates the use of inverse kinematics and simple geometric constraints as tools for the animator. Previous applications of inverse kinematic algorithms to computer animation are reviewed. A pair of alternative algorithms suitable for a direct manipulation interface are presented and qualitatively compared. Application of these algorithms to enforce simple geometric constraints on a figure during interactive manipulation is discussed. An implementation of one of these algorithms within an existing figure animation editor is described, which provides constrained inverse kinematic figure manipulation for the creation of keyframes.

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Chapter 1

Introduction

Computer graphics has advanced to a point where generating images of striking realism and complexity has become almost commonplace. However, making objects move convincingly within these pictures remains difficult, particularly as object models grow increasingly complex. The specification and control of motion for computer animation has emerged as one of the principal areas of research within the computer graphics community.

One area in particular which continues to receive attention is that of figure animation. The goal of work in this area is to provide a means of generating life-like, possibly human-like, articulated figures, and to design and control their actions within simulated environments. Animated human figures could, for example, be placed in simulated environments for ergonomic evaluation, or simply to provide some aesthetic qualities to a presentation. In the arts and entertainment area, the concept of computer-generated characters roaming through artificial worlds seems universally appealing.

Although figure animation raises technical challenges in both modelling and rendering, the fundamental problem of designing and controlling movement for these figures remains a particularly difficult one. Part of the problem lies in deciding from which level of detail to approach the task. At one end of the scale, the movements of individual parts of the body must be known for each instant in time. At the other end of the scale, coordinating movements and handling interaction between figures and with the environment may require algorithms based on behavioural rules and knowledge bases. Many of the most impressive examples of figure animation by computer have been the result of algorithms implementing high-level behavioural and motor control models. However, these algorithms are often limited to generating specific, usually repetitive, movement patterns such as walking and running. For the animator who wishes to create *new* movements, there is little alternative to painstakingly constructing the movement by hand. Given the complex structure of a typical articulated figure, this can involve an inordinate amount of work.

The motivation behind this work is a desire to improve the animation capabilities of an existing interactive articulated figure animation package, which is currently used to create movements for both dance and animation. It is shown how *inverse kinematic* techniques for controlling robotic manipulators can be adopted to relieve the animator of some of the more tedious aspects of creating new movements by hand. After reviewing the inverse kinematic problem and solutions that have previously been applied to figure animation, a pair of alternative solution algorithms are presented and qualitatively compared. These algorithms are simple, yet effective, and can support both direct manipulation of articulated figures as well as the imposition of simple geometric constraints upon a figure. Implementations of these algorithms are presented, and are applied to develop a basic set of interactive tools for figure manipulation and animation.

1.1 Organization

Chapter 2 reviews computer animation techniques in general, and discusses their applicability in the context of figure animation. In Chapter 3 the inverse kinematic problem is stated, and common approaches to solving the problem are reviewed. In Chapter 4 a pair of fast, reliable inverse kinematic algorithms are described, suitable for interactive manipulation tasks and differing from previous algorithms adopted for computer graphics. In Chapter 5 procedures for satisfying simple geometric constraints using these algorithms are considered. Chapter 6 introduces an interactive figure animation editor and discusses implementation of the algorithms as positioning aids.

Chapter 2

Approaches to Figure Animation

Placing this work in context requires some understanding of computer animation techniques in general, and of how they may be applied to figure animation in particular. This chapter provides an overview of the advantages and disadvantages of basic motion control techniques for figure animation.

The emphasis here is on methods to create and control the movements of articulated figures, rather than simply replaying digitized movement. It is fair to say that for many productions, digitizing, or *rotoscoping*, the movements of real subjects remains the method of choice for obtaining convincing life-like motion. Rotoscoping can refer to techniques ranging from visually matching graphic images to prerecorded video footage, to attaching some sort of sensors to a performer's body, whose positions can be tracked by computer and stored for later playback. Neither of these are particularly attractive options: the former being quite tedious, and the latter relying on the availability of reliable, unobtrusive instrumentation for the body, and sophisticated software to reconstruct the original motion from the sensor data, neither of which are readily available yet. A further limitation of rotoscoping is that a figure animated in this way is limited to those movements actually performed by a live subject. Computer animation techniques can be applied to animate figures in situations for which rotoscoping is neither a viable nor practical solution.

2.1 Body Models

2.1.1 Scope

First we must decide exactly what we are trying to animate. Although the ideal computer-generated "character" would include muscle and tissue that deforms during movement, skin and clothing that wrinkles and stretches, hair that flows, and expressive facial features, the accurate modelling, animation, and rendering of these attributes are research topics in their own right, and work in these

areas is still at the experimental stage. For the time being we will have to restrict our attention to animating simple approximations to real bodies. It is useful to think of these simple approximations as a skeletal layer, upon which muscle, tissue and skin can later be layered. The important point here is that any body model can be animated by moving an underlying skeletal approximation, which need not bear any resemblance to the final rendered appearance of the character. Thus the motion control problem for figures reduces to that of controlling the movement of an abstract articulated skeleton.

2.1.2 Skeleton Modelling

A skeleton can be represented by a collection of simple rigid objects connected together by joints. The joints are usually rotational, but may also be sliding (or prismatic). Each rotary joint may allow rotation in 1, 2, or 3 orthogonal directions; these are the degrees of freedom (DOF) of the joint. A detailed approximation to the human skeleton may have as many as 200 degrees of freedom, although often fewer suffice. Restrictions on the allowable range of movements for a joint can be approximated by limiting the rotation angle in each of the rotation directions at each joint.

The individual objects comprising the skeleton are each defined in their own local coordinate systems, and are assembled into a recognizable figure in a global *world* coordinate system by a nested series of transformations. In Figure 2.1 a simple articulated limb is built up by applying local rotations and translations to blocks defined in their own local coordinate systems.

More complex skeletons can be built up by arranging the segments in a tree-structured hierarchy. Each node in the tree maintains the rotations currently in effect at the corresponding joint; these joint rotations are offsets from the orientation of the parent segment in the tree. These nested transformations in the hierarchy ensure that segments inherit the rotations applied to joints higher in the tree; a rotation applied at, say, the shoulder joint, causes the entire arm to rotate, and not just the upper arm segment. One joint in the skeleton needs to be specified as the root of the tree; transformations applied to this joint move the entire skeleton in the world coordinate system. The choice of which joint is to serve as the root is irrelevant, and it is convenient to be able to restructure an existing hierarchy around a new root joint at any time. The global transformations applied to any particular object within the skeleton can be computed by traversing the hierarchy from the root to the segment and concatenating the local transformations at each joint visited by the traversal.

Most animation systems provide a means of building up the transformation hierarchy needed to define a skeleton, and it is easy enough to define a simple grammar for specifying skeletons [Zel82b]. Sims' [SZ88] has described an interactive editor for designing new skeletons which applies some simple heuristics to streamline the process. Regardless of how it is created, a skeleton definition will

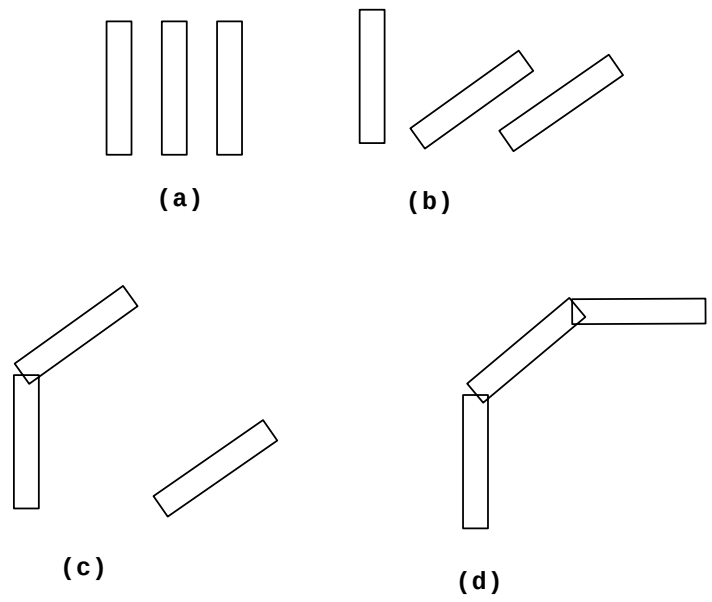


Figure 2.1: (a) 3 objects defined in local coordinate systems. (b) Local rotations applied. (c-d) Local translations applied.

minimally specify the individual body segment lengths, the joint degrees of freedom, and the overall hierarchy of the structure.

A skeleton can be animated by varying the local rotations applied at each joint over time, as well as the global translation applied at the root joint. The motion specification and control problem is that of managing the way in which these transformations change over time. In general, there are two fundamental approaches to this problem: *kinematic* and *dynamic*. The following sections review both kinematic and dynamic methods for motion specification in general, the types of control available for each, and applications of these to figure animation in particular.

2.2 Kinematic Methods

2.2.1 Forward Kinematics

Forward kinematics involves explicitly setting the position and orientation of objects at specific frame times. For skeletons, this means directly setting the rotations at selected joints, and possibly the global translation applied to the root joint, to create a *pose*. To avoid doing this for each frame of an animation, a series of *keyframe* poses can be specified at different frames, with intermediate

poses calculated by interpolating the joint parameters between the keyframes. The figure can then be animated by displaying each intermediate pose.

While linear interpolation between keyframes is the simplest method for generating these intermediate poses, the resulting motion is usually unsatisfactory. Discontinuous first derivatives in the interpolated joint angles at the keyframes lend a jerky, robotic quality to the motion. The use of higher-order interpolation methods, such as piecewise splines, can provide continuous velocity and acceleration, and hence smoother transitions between and through the keyframes. Keyframe interpolation is well established [Ste83] [Gom85] [HS85] [Sho85] [Stu87], and is invariably provided in commercial animation systems.

Controlling interpolation

Interpolation often produces intermediate values that do not quite meet the animator's requirements; some control over the interpolation process is crucial [Las87]. The interpolated values for a single DOF over the course of an animation form a *trajectory* curve, which (usually) passes through the keyframe values. The shape of the trajectory, and hence the motion of the object, is dependant on both the keyframed values and the type of interpolating spline used. An interactive editor which allows the animator to view and modify the shape of a trajectory can be a useful tool. Once a trajectory is defined, the quality of the movement can be further modified by varying the rate at which the trajectory is traversed. A number of parameterized interpolation methods have been proposed which provide varying degrees of control over both the shape of a trajectory and variations in speed along the trajectory.

Kochanek [KB84] describes an interpolation technique based on a generalized form of piecewise cubic Hermite splines. Three parameters - *continuity*, *tension*, and *bias* - are provided to control the length and direction of vectors tangent to the trajectory at the keyframes. Modifying the direction of the tangent vectors gives local control over the shape of the curve as it passes through a keyframe. Changing the length of the tangent vectors affects the rate of change of the interpolated value around the keyframe, and thus provides some control over speed. Some traditional animation effects such as *action follow-through* and *exaggeration* [Las87] can be achieved with appropriate settings of these parameters. Unfortunately, since all three parameters in the spline formulation affect the shape of the curve, the method provides no means for modifying speed along a trajectory without modifying the trajectory itself.

Steketee and Badler [SB85] advocate a double-interpolant method which does separate timing control from the trajectory itself. As before, a trajectory is defined as a piecewise cubic spline passing through a series of keyframed values. An additional spline curve is also introduced to control the

parameter with which the trajectory curve is sampled. This provides control over the *parametric* speed at which the trajectory curve is traversed. However, there is often no meaningful relationship between parametric speed and actual speed in the geometric sense; sampling a curve at uniformly spaced parametric points will not necessarily yield uniformly spaced points in space. This approach to timing adjustment, therefore, is somewhat *ad hoc* and non-intuitive, requiring a trial-and-error process on the part of the animator to achieve the desired velocity profile along the trajectory.

More intuitive control over speed along a trajectory can be obtained by reparameterizing the trajectory curve by arc-length. Arc-length parameterization provides a direct relationship between parametric speed and geometric speed along a trajectory; the distance Δd travelled along a trajectory is proportional to the increment Δs of the trajectory's arc-length parameter s. Allowing the animator to sketch a curve representing s over time provides an intuitive mechanism for varying speed along the trajectory [BH89]. However, although theoretically a reparameterization by arc-length exists for any curve, it is often not possible to find an analytic solution for arbitrary curves, and one must resort to numerical approximation methods [Gir86] [GP90].

Evaluation

Keyframe-based computer animation has a direct analogy in traditional animation, where key animation cels are drawn by senior animators, while less experienced animators draw the action in the intermediate cels. Computer-based keyframing is intuitive, and the interpolation can usually be performed fast enough to provide near real-time feedback. For skeleton animation, however, keyframe interpolation does not work well; the few good examples of keyframed figure animation are more a tribute to the skill and patience of the animator than to the technique's suitability for the task.

One major difficulty can be labelled the "degrees of freedom" problem: for interesting skeletons, there are simply too many DOFs for which values must be provided; the level of detail required from the animator to specify even a single key pose is excessive. Trying to control the interpolated motion by manually modifying possibly hundreds of trajectory curves can be tedious, frustrating and error-prone. While it is essential that the animator have some control at the joint level, higher levels of control are desirable for specifying the coordinated movements of groups of joints.

Even supposing that the number of degrees of freedom within a figure is manageable, the common practice of displaying interpolated joint angles as a set of three splined trajectory curves [1] is rarely helpful. Unlike translations, an ordered series of rotations do not combine intuitively, making it difficult to predict the consequences of editing a single rotation trajectory and almost impossible to decide on the appropriate changes to all three curves which will produce a desired change in a single

[1] one each for the X,Y, and Z rotation directions at each joint

body segment's motion.

The hierarchical structure of the skeleton also causes problems. The only joint which an animator can explicitly position is the root joint in the hierarchy; the positions of all other objects in the skeleton depend on the rotations at ancestor joints. This makes it difficult to enforce positional constraints when creating a keyframe pose. For example, if the hierarchy root for a biped is at the pelvis, then placing a foot on the floor and keeping it there is troublesome; if the foot is already in place, then a bend at the knee will move the foot, which must then be repositioned by modifying the rotation at the hip joint. The ability to rearrange the hierarchy about a new root joint is only marginally useful. In our example, making the support foot the new root of the hierarchy would allow a knee bend which leaves the foot in place. However, this will also move the rest of the body, which may move another, previously positioned body segment, such as the other foot. This makes enforcing multiple positional constraints a frustrating process.

The same problem crops up during interpolation. Even if an animator has made sure that both feet are positioned correctly in a series of keyframe poses, there is no guarantee that simply interpolating joint rotations will maintain the correct foot positions at the intermediate frames. It is quite common to see interpolated keyframed sequences for figures in which the feet seem to penetrate through, or slide around on, the floor. While this can be remedied by specifying additional keyframes, as the keyframe spacing becomes smaller the animation process begins to resemble the frame-by-frame positioning of traditional stop-action animation (claymation, for example). This defeats the whole purpose of interpolation, which is intended to relieve the animator from the tedium of specifying the motion on a frame-by-frame basis.

While forward kinematics combined with a simple interpolation scheme may suffice for animating simple objects, it is not really up to the task of animating articulated figures.

2.2.2 Inverse Kinematics

Using forward kinematics, the position of any object within a skeleton can only be indirectly controlled by specifying rotations at the joints between the root and the object itself. In contrast, *inverse kinematic* techniques provide direct control over the placement of an *end-effector* object at the end of a *kinematic chain* of joints, solving for the joint rotations which place the object at the desired location. In light of the preceeding discussion, it should be apparent that inverse kinematics offers an attractive alternative to explicitly rotating individual joints within a skeleton. An animator can instead directly specify the position of an end-effector, while the system automatically computes the joint angles needed to place the part. Not surprisingly, the inverse kinematic problem has been studied extensively in the robotics field, although it is only fairly recently that the techniques have

been adopted for computer animation.

Chadwick's *Critter* system permits inverse kinematic manipulation of a skeleton for creating keyframes [CHP89]. Badler has proposed an inverse kinematic algorithm to enforce positional constraints on multiple body parts during skeleton manipulation [BMW87], and has incorporated joint range limits into the inverse kinematic solution [CP90, ZB89]. Both Girard's *PODA* system [GM85] and Sims' gait controller [SZ88] provided high-level locomotion models for skeletons, using inverse kinematics to generate the leg motion. In these systems, a planning stage determines foot placements and trajectories, while the inverse kinematic algorithm is responsible for generating the leg joint angles as the feet are moved along trajectories between each foot-hold.

Inverse kinematics provides higher-level control over joint hierarchies than simple forward kinematics; moving the limbs of a skeleton becomes much more manageable. However, often the underlying method for generating motion still relies on strictly kinematic methods. Unfortunately, kinematic methods do not produce convincing movement without a considerable amount of effort on the animator's part. Often, the motion exhibits a weightless quality which is difficult to dispel by editing the trajectories and timing for individual degrees of freedom. Kinematic methods, both forward and inverse, do not produce movement with the sort of *dynamic integrity* we have come to expect from our experience with the physical laws of the real world.

2.3 Dynamic Methods

Animation based on dynamic simulation is attractive because the generated motion adheres to physical laws, providing a level of realism that is extremely difficult to duplicate with kinematic methods. For dynamic analysis, object descriptions must include such physical attributes as the center of mass, the total mass, and the moments and products of inertia. Although there are many formulations for the equations of motion, they are all essentially equivalent to the familiar $F = ma$, which relates the acceleration a an object of mass m undergoes in response to a force F applied at the object's center of mass [2]. The motion generated by physical simulation is controlled by the application of forces and torques, which may vary over time. Techniques for dynamic motion control can be categorized as either *forward dynamic* methods or *inverse dynamic* methods. The essential distinction between the two is in the way that the basic forces and torques driving the motion are arrived at.

2.3.1 Forward Dynamics

Forward dynamics involves explicit application of time-varying forces and torques to objects. Some forces, such as those due to gravity and collisions between objects, may be handled automatically

[2] a similar equation relates angular acceleration to applied torques

by the animation system; other forces are applied directly by the animator to objects in the scene. The motion is approximated by taking a series of discrete steps in time, and at each step solving the equations of motion for the acceleration an object undergoes in response to the applied forces. Given the position and velocity of an object from the previous time step, the acceleration a can be twice integrated to determine a new velocity and position, respectively, for the current time step. A good introduction and overview of the basics of forward dynamic simulation for animating rigid bodies can be found in [Wil91]. A comprehensive approach to simulating the motion of rigid polyhedral objects, accounting for collisions, is presented by Hahn [Hah88].

Extending this approach to the simulation of articulated skeletons is challenging. In general, there will be one equation of motion for each degree of freedom in the skeleton. This leads to a large system of equations, which must be solved by numerical methods at considerable computational expense. The formulation adopted to represent the equations of motion significantly affects the cost of the solution method. A solution for the matrix-based Gibbs-Appell formulation, for example, has O(n^4) complexity for n degrees of freedom [Wil87]. Armstrong has proposed an alternative recursive formulation which reduces the complexity to $O(n)$ [AG85], enabling dynamic simulations of simple articulated structures to be performed in close to real-time. But dynamic simulation of reasonably complex articulated skeletons cannot in general be performed at interactive speeds on single-processor machines, although the recursive formulation may be fast enough to be tolerable.

Complicating matters is the fact that the equations of motion for articulated skeletons are considerably more complex than those for simple objects, since they must include terms to model the interactions between connected body parts. This coupling of the dynamics equations makes control extrememly difficult, since movement of one segment of the skeleton will exert forces and torques on adjacent segments; the notion that the motion of the skeleton can be adequately controlled by applying joint torques individually is incorrect [Wil86]. Efforts to counteract this unwanted propagation of torques usually involve placing springs and dampers at each joint to maintain a desired orientation. Unfortunately, this type of control invariably leads to a *stiff* set of equations, which causes severe instability in most numerical solution techniques. A summary of numerical stability and control issues that must be addressed during dynamic simulation is presented in [Gir91].

Compounding the problem of numerical instabilities is the fact that the equations of motion for articulated skeletons are inherently ill-conditioned, independent of their formulation [Mac90]. The ill-conditioning arises when the skeleton assumes a posture in which small incremental changes in one degree of freedom produce large accelerations elsewhere; almost all numerical solution techniques have difficulty handling such cases. Maciejewski contends that these situations occur frequently for articulated figures, and are inherent in the structure of most skeletons. The ill-conditioning of the

equations has implications not only for dynamic analysis, but inverse kinematic algorithms as well; [Mac90] gives a lucid description of the problem, and discusses methods for detecting and handling the ill-conditioning in both cases.

One of the earliest attempts to control an articulated figure purely through forward dynamic simulation was Wilhelms' *Virya* system [Wil86]. *Virya* permitted the interactive design of force or torque versus time functions for individual degrees of freedom. Force and torque keyframes could be specified at different times; cubic splines were then used to construct the force and torque profiles over the course of the entire motion sequence. During dynamic simulation, these force/torque profiles were sampled, and combined with forces due to collisions and gravity, to determine instantaneous force and torque measurements for the current time step. The use of interpolating curves is conceptually similar to the direct kinematic keyframe interpolation approach described previously. The difference is that the motion is driven not directly by the interpolated curves, but indirectly through the equations of motion. *Virya* exhibited most of the problems outlined above. In particular, Wilhelms reports that the coupling of the dynamic equations made control of the figure difficult and non-intuitive. Other efforts to simulate skeleton motion using pure forward dynamics report similar problems [AG85] [WCH88].

Even supposing that a reliable and fast numerical solution technique is available, the lack of intuitive control remains the principal problem in using forward dynamics for animation. In fact, forward dynamic simulation is best suited for tasks which can be posed as initial-value problems. That is, tasks for which initial positions and velocities, and force/torque profiles, are known *a priori*, and the goal is to generate the resulting motion. This formulation may be satisfactory for animating scenes of simple inanimate objects realistically tumbling and bouncing through an environment, but does not apply for the animation of specific tasks. For example, simulating a ball bouncing on a floor is simple to do given an initial height and velocity; the simulation need only consider the force of gravity, and reactions to collisions with the floor, to generate convincing motion. However, if the goal is to have the ball bounce three times and land in a cup the problem is much more difficult; the exact initial position and release velocity of the ball which will land it in the cup is difficult to determine. Yet this is precisely the sort of problem that appears in animation; the animator knows what motion should occur, but does not know in advance the initial conditions and force/torque profiles needed to produce the desired result.

2.3.2 Inverse Dynamics

Inverse dynamic methods automatically determine the force and torque functions needed to accomplish a stated goal. In the degenerate case, the stated goal is a complete description of the motion,

and the aim is to determine the forces and torques which reproduce the motion under forward dynamic simulation. While this case is of interest in robotics, its application is of little use in an animation system; after all, if the motion trajectories and timing are known beforehand the expense of the physical simulation is unnecessary. More interesting are recent methods which allow relatively high-level constraints or goals to be specified, and which then compute the forces and torques necessary to meet the goals.

Geometric Constraints

Barzel and Barr [BB87] made early use of inverse dynamics for modelling. A model was defined as a collection of objects related by geometric constraints. A number of useful simple constraints for modelling were presented, including *point-to-point* constraints for attaching two objects together, *point-to-path* constraints for moving an object along a predefined path, and *twist* constraints to control an object's orientation. The constraints were used to introduce forces and torques into a forward dynamics simulation of the model. These constraint forces and torques act in concert to move the model towards a state in which all the constraints are satisfied. This approach blurs somewhat the distinction between modelling and motion control, as it allows for the animation of self-assembling structures; if the constituent parts of the model initially are in a state which violates the geometric constraints, turning on dynamic simulation results in the model assembling itself using the laws of Newtonian mechanics.

Forsey and Wilhelms have used inverse dynamics to manipulate an articulated skeleton into keyframe positions for a traditional kinematic interpolation system [FW88]. The *Manikin* system performed dynamic analysis during interaction with the figure, using Armstrong's recursive formulation for the equations of motion. A positional goal for a body part could be specified interactively, with *Manikin* computing the forces to push the part towards the goal. This allowed manipulation of the figure in a manner similar to inverse kinematic manipulation. The imposition of positional constraints upon body parts was accomplished by artificially increasing the mass of constrained parts, with the system constantly computing additional forces necessary to keep the part in place as other parts were moved. Motion sequences could be generated by storing the state of the body at different points during the dynamic analysis, and later using these stored states as keyframes for kinematic interpolation.

The *penalty-force* approach taken here, of converting all constraints into forces and torques which steer the motion during dynamic analysis, has its limitations. The penalty forces are often modelled as simulated springs and dampers, which deliver a force proportional to the velocity of the motion. This method of control is vulnerable to stiffness in the resulting system of equations, and by undesirable oscillations about constraint satisfaction points. Choosing appropriate spring and damping

coefficients for the constraints is often a matter of trial-and-error.

In contrast to the penalty-force approach, a number of formulations for the dynamic equations of motion can include explicit constraint equations. Isaacs' *Dynamo* system [IC87] [IC88] combines keyframed kinematic constraints with inverse dynamics. Rather than causing the introduction of additional forces into the simulation, the kinematic constraints are instead used to remove degrees of freedom from the system, since they implicitly specify some of the accelerations in the system. The remaining accelerations for unconstrained DOFs can then be solved for. The solution method ensures that reactant forces due to the keyframe constraints are introduced into the solution for the unconstrained DOFs. This allows the kinematic constraints to specify motion for some parts of a skeleton, while the other unconstrained parts react realistically to the prescribed motion. In cases where all parts are constrained, the technique reduces to a simple keyframing approach. This approach illustrates an interesting mixture of dynamic simulation with kinematic control. However, Isaacs' most ambitious attempt at skeleton animation is the simulation of a traditional marionette controlled by rods and strings attached to the limbs. While technically impressive, this example points out the need for better methods of control over dynamically simulated skeleton motion.

Non-Geometric Constraints

Consider the inverse dynamic problem of moving a point mass from position A to position B in a given time interval t. There is no unique force function over the interval t which will accomplish this; the system must choose between applying a large force for a short period of time, or applying a smaller force over a longer period - both methods may achieve the goal of reaching the keyframed position B at time t. This problem is one of determining not only what is to occur, in this case moving from A to B, but also how the motion is to occur. A number of methods have been proposed which attempt to describe the quality of motion by considering non-geometric constraints in the inverse dynamic solution. These approaches are based on well-established techniques for optimizing functions subject to a set of constraints.

Brotman and Netravali [BN88] propose an inverse dynamic approach to motion interpolation which uses penalty forces to enforce keyframed kinematic constraints. However, the solution incorporates an additional constraint on the energy exerted by these penalty forces. The problem is formulated as that of solving for the set of constraint forces which minimizes the energy expended in meeting the constraints imposed by kinematic keyframe values.

Girard [Gir91] has applied constrained optimization techniques to determine speed distribution along predefined limb trajectories for articulated figures. Girard notes that the choice of optimization criteria has a significant effect on the perceived quality of motion. Solving for a velocity profile which

minimizes energy expenditure yields a relaxed swinging motion for the limb, while minimizing jerk about the end of the limb yields movement suggestive of such goal-directed tasks as reaching for an object. The establishment of additional correspondences between optimization criteria and expressive qualities of movement remains an open area of research.

These constrained optimization methods assume that the complete or partial motion paths for limbs are known in advance, and attempt to derive the "best" set of forces and torques which move the limb along the path. This side-steps the fundamental problem of synthesizing the limb trajectories for coordinated movement in the first place. Witkin and Kass [WK88] have proposed an intriguing method of motion synthesis they call "Spacetime Constraints", which they demonstrated to be capable of synthesizing both the trajectories and the timing of movements for simple articulated figures. This use of constrained optimization seems particularly promising, as it seems capable of producing complex, coordinated, physically-correct motion with very little input from a user. However, the approach results in very large systems of equations which must be solved, and cannot be considered useful for interactive figure animation. Ongoing research is addressing the interactivity limitations of the method [Coh92].

Badler [LWZB90] has used a form of constraint-based inverse dynamics to synthesize the trajectories of limbs charged with the task of moving a load between two different positions. The trajectories are computed incrementally, and are constrained by measures of strength, comfort, and exertion. The iterative nature of the algorithm differs fundamentally from the global solution found by optimization methods. Instead, a set of biomechanical heuristics, which are intended to mimic the process by which people move loads, are used to guide the solution process. The method successfully produces feasible, albeit sub-optimal, limb trajectories which accomplish the task.

2.4 Control Issues

The research efforts outlined above are attempts to provide higher levels of control over both kinematic and dynamic motion. The goal is to be able to specify movements at the *task* level, and to have the system take care of the underlying details of producing the motion. Given the current state of these efforts, it seems that it will be some time before the emergence of systems capable of synthesizing motion to accomplish arbitrary tasks. However, there has been some success in developing special purpose control strategies for specific types of movements. The system is responsible for decomposing high-level task descriptions, such as "walk to the door" or "reach for the cup", into lower-level movement primitives, and for the coordination of these primitives. The low-level primitives may consist of keyframes for interpolation, inverse kinematic goals, forward dynamic simulations, constrained inverse dynamic goals, or a mixture of all these approaches.

Zeltzer was an early proponent of the need for high-level control over articulated figures [Zel82a]. He describes a control strategy for synthesizing walking sequences for a skeleton. High-level walking instructions are decomposed into a set of *motor control programs* (MCP), which drive the motions of individual limbs or joints. The control stategy is based on a finite-state machine responsible for activating and deactivating the appropriate MCPs at the appropriate times. Zeltzer's MCPs consisted of kinematic joint values obtained from rotoscoped human walks, and thus were purely kinematic. Nevertheless the system demonstrated the usefulness of the concept.

Building on Zeltzer's work, Bruderlin [BC89] developed a similar hierarchical control strategy for generating biped walking sequences, but incorporated dynamic simulation to derive leg motion, rather than relying on rotoscoped data. The user is able to instruct a skeleton to walk at a particular speed, and is able to specify both desired step frequency and step length. These instructions are decomposed into dynamically-based low-level MCPs which drive the motion of an abstract, kneeless pair of legs. The MCPs essentially perform dynamic interpolation of a set of kinematic keyframes for the leg movements during the walk cycle. The kinematic keyframe values and spacing are derived from the input parameters, combined with knowledge about human locomotion patterns gleaned from the biomechanics literature. The forces and torques driving the motion of the simplified walking mechanism are iteratively adjusted until the keyframed joint angles are achieved at the correct times. A purely kinematic overlay of the skeleton's jointed legs onto the underlying mechanism is then performed. The algorithm is able to produce a wide range of realistic walking sequences, and is a true hybrid of both dynamic and kinematic motion control. The decision to use a simplified dynamic model specifically tuned for walking seems sound; the resulting system of equations is small, relatively stable, and inexpensive to solve. A similar approach has been used by this author to build a jumping algorithm based on the simulation of a simple underlying mass-and-spring model.

Unfortunately, the high-level control provided by algorithms of this nature come at the expense of generality; each control strategy must be tuned for a specific movement. But developing such a control strategy is difficult. Deriving the equations for simulating the dynamics of the underlying mechanism requires some mathematical sophistication. In the absence of an inverse kinematic algorithm, Bruderlin's method of mapping the motion of the underlying dynamic model to the motion of the skeleton can pose problems to the implementor. To a large extent, the success of the above control strategies is due to the predictable, repetitive nature of locomotion. Developing high-level control strategies for arbitrary movement sequences still seems a distant goal.

2.5 Summary

What has hopefully emerged from the discussion so far is that no one technique has emerged as a clear winner. A successful figure animation system is likely to incorporate all of the techniques discussed above, to some degree. Pure dynamics applied to figure animation seems to raise as many problems as it addresses, unless it is confined to specific movement control strategies. The research into automatic motion synthesis from high-level constraints, while promising, is still at too early a stage to be considered useful. For the time being, designing arbitrary movement sequences remains in the hands of the animator.

A simple interactive keyframe editor in the hands of a user who understands how the body moves, and has some patience, can produce some surprisingly good animation sequences, even for figures as complex as the human form. Dynamic simulation or algorithmic motion models, while useful in some contexts, will only be appreciated if they can alleviate some of the work involved in interactively hand-crafting new movement sequences, and it can be argued that given the current state of research in these areas this is not generally the case. The most promising interactive techniques reviewed in this chapter are those based on the use of inverse kinematics, which provide a level of control higher than simple forward kinematics yet still leave the user with complete control over the animation.

In the remaining chapters, the use of inverse kinematics to complement an existing interactive keyframe editor is explored. The goal is to address the limitations of a simple forward kinematic approach, by providing a set of tools which support direct manipulation of kinematic chains within a figure, and the imposition of simple geometric constraints which are maintained during keyframe creation and interpolation. Along the way we identify two inverse kinematic algorithms which differ from those previously adopted for computer graphics, and describe their suitability to the problem.

Chapter 3

Inverse Kinematics

The inverse kinematic problem has been studied extensively in the robotics literature, which remains the best source of information on the subject. In this chapter we formally state the problem and review the most common approaches to solving it. Previous applications of these approaches to computer graphics are also described.

3.1 The Inverse Kinematic Problem

Section 2.1.2 showed that a skeleton can be modelled as a hierarchical collection of rigid objects connected by joints. We will refer to a kinematic chain of segments within a skeleton as a *manipulator*, and will assume that the joints connecting segments within this chain are revolute joints rotating about a single axis. One end of the manipulator, the *base*, is fixed and cannot move; the distal end of the chain is free to move. The *end-effector* is embedded in the coordinate frame of the most distal joint in the chain; the end-effector position is a point within this frame and the end-effector orientation refers to the orientation of the frame itself.

At each joint in the chain a joint variable determines a transformation $\mathbf{M}$ between the two adjacent coordinate frames sharing the joint. The transformation $\mathbf{M}_i$ at a rotation joint i is a concatenation of a translation and a rotation, both of which are relative to the coordinate frame of joint i's parent. That is,

$$\mathbf{M}_i = \mathbf{T}(x_i, y_i, z_i)\mathbf{R}(\theta_i) \tag{3.1}$$

where $\mathbf{T}(x_i, y_i, z_i)$ is the matrix that translates by the offset of joint i from its parent joint $i-1$, and $\mathbf{R}(\theta_i)$ is the matrix that rotates by θ_i about joint i's rotation axis.

The relationship between any two coordinate systems i and j in the chain is found by concatenating the transformations at the joints encountered during a traversal from joint i to joint j:

$$\mathbf{M}_i^{\ j} = \mathbf{M}_i \mathbf{M}_{i+1} \cdots \mathbf{M}_{j-1} \mathbf{M}_j \tag{3.2}$$

So the position and orientation of the end-effector with respect to the base frame is found by simply concatenating the transformations at each joint in the manipulator.

Given a vector $\mathbf{q}$ of known joint variables, then, the *forward kinematic* problem of computing the position and orientation vector $\mathbf{x}$ of the end-effector, is a simple matter of matrix concatenation, and has the form

$$\mathbf{x} = \mathbf{f}(\mathbf{q}) \tag{3.3}$$

But if the goal is to place the end-effector at a specified position and orientation $\mathbf{x}$, then determining the appropriate joint variable vector $\mathbf{q}$ to achieve the goal requires a solution to the inverse of (3.3),

$$\mathbf{q} = \mathbf{f}^{-1}(\mathbf{x}) \tag{3.4}$$

Solving this *inverse kinematic* problem is not so simple. The function $\mathbf{f}$ is nonlinear, and while there is a unique mapping from $\mathbf{q}$ to $\mathbf{x}$ in equation (3.3), the same cannot be said for the inverse mapping of (3.4) - there may be many $\mathbf{q}$'s for a particular $\mathbf{x}$. The most direct approach for solving the problem would be to obtain a closed-form solution to (3.4). But closed-form solutions can only be derived for a restricted set of manipulators with specific characteristics, and even these result in a set of non-linear equations to be solved[Pau81]. A general analytic solution for arbitrary manipulators does not exist; instead the problem must be solved with numerical methods for solving systems of non-linear equations. The most common solution methods are based on either matrix inversion or optimization techniques.

3.2 Resolved Motion Rate Control

Since the non-linear nature of equation (3.4) makes it difficult to solve, a natural approach is to linearize the problem about the current manipulator configuration - then the relationship between joint velocities and the velocity of the end-effector is

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{3.5}$$

The linear relationship is given by the *Jacobian* matrix

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \tag{3.6}$$

which maps changes in the joint variables $\mathbf{q}$ to changes in the end-effector position and orientation $\mathbf{x}$. $\mathbf{J}$ is an $m \times n$ matrix, where n is the number of joint variables and m is the dimension of the end-effector vector $\mathbf{x}$, which is usually either 3 for a simple positioning task, or 6 for a more general position-and-orientation task. The ith column of $\mathbf{J}$ represents the incremental change in the position (and orientation) of the end-effector resulting from an incremental change in the joint variable q_i.

Inverting the relationship of (3.5) provides the basis for *resolved motion rate control*

$$\dot{\mathbf{q}} = \mathbf{J^{-1}}(\mathbf{q})\dot{\mathbf{x}} \tag{3.7}$$

If the inverse of $\mathbf{J}$ is known, we can compute incremental changes in the joint variables which produce a desired incremental change in the end-effector position and orientation.

A simple iterative scheme for solving the inverse kinematic problem can be based on equation (3.7). At each iteration a desired $\dot{\mathbf{x}}$ can be computed from the current and desired end-effector positions. The joint velocities $\dot{\mathbf{q}}$ can then be computed using the Jacobian inverse, and integrated once to find a new joint state vector $\mathbf{q}$. The procedure repeats until the end-effector has reached the desired goal. Note that since the linear relationship represented by $\mathbf{J}$ is only valid for small perturbations in the manipulator configuration, $\mathbf{J}(\mathbf{q})$ must be recomputed at each iteration. A procedure for efficiently computing the Jacobian is presented in Section 4.1.3.

Of course, this scheme assumes that the Jacobian matrix is invertible; that $\mathbf{J}$ is both square and non-singular. This assumption is not, in general, a valid one. Difficulties arise when a manipulator is *redundant*, or when it passes through or near a *singular* configuration.

3.2.1 Redundancy

A manipulator is considered *kinematically redundant* when it possesses more degrees of freedom than are required to specify a goal for the end-effector. For example, consider the simple 2D case in Figure (3.1). The manipulator possesses 3 degrees of freedom: the rotation angles at each joint. For a simple positioning task, the goal is to place the end-effector (the tip of the distal link of the chain) at some point (x, y). As the figure shows, for a given goal (x, y) there is no unique solution; each of the configurations shown will place the tip at the goal position. The manipulator is therefore redundant for this 2D positioning task.

In general, positioning an object in Cartesian space requires the specification of six coordinates: three for location and three for orientation. Therefore, any manipulator possessing more than six degrees of freedom is redundant for the general 3D-space positioning task, and there is no unique

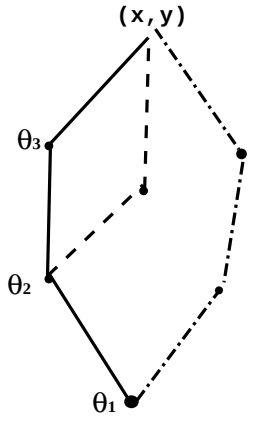


Figure 3.1: Three configurations of a 2D redundant manipulator

set of joint values solving the inverse kinematic problem.

For a redundant manipulator, the Jacobian matrix has fewer rows than columns, and cannot be inverted. In this case, equation (3.7) is under-determined, and there are an infinite number of solutions from which to choose. If $\mathbf{J^{-1}}$ in (3.7) is replaced by some generalized inverse $\mathbf{J}^\dagger$, then a useful solution to the under-determined problem can be found. One such generalized inverse is the Moore-Penrose *pseudoinverse* [Gre59, GM85]. It can be shown [KH83] that this pseudoinverse is optimal in the sense that it yields solutions with a minimum Euclidean norm for cases in which (3.7) is under-determined ($m < n$), and that in cases in which the system is over-determined ($m > n$) a least-squares solution is obtained. In practice, these properties ensure that joints move as little as possible to match the desired end-effector velocity as closely as possible.

Exploiting Redundancy

Since a redundant manipulator can satisfy a positioning task in any number of ways, it is often useful to consider exploiting the redundancy in an attempt to satisfy some secondary criteria. This can be accomplished by incorporating an additional term into equation (3.7)

$$\dot{\mathbf{q}} = \mathbf{J}^\dagger \dot{\mathbf{x}} + (\mathbf{I} - \mathbf{J}^\dagger \mathbf{J}) \nabla H(\mathbf{q}) \tag{3.8}$$

The function $H(\mathbf{q})$ is a measure of some criterion to be minimized, subject to satisfying the primary positioning task. The other component, $(\mathbf{I} - \mathbf{J}^\dagger \mathbf{J})$, is a projection operator which selects those components of the gradient vector $\nabla H(\mathbf{q})$ which lie in the set of *homogeneous* solutions to (3.5). A homogeneous solution to (3.5) is a set of joint velocities $\dot{\mathbf{q}}$ which does not change the end-effector

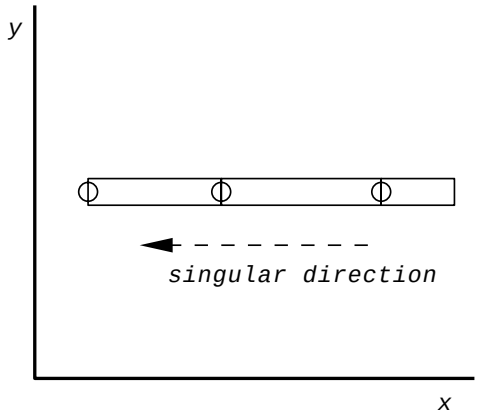


Figure 3.2: A manipulator in a singular configuration

position (i.e. for which $\dot{\mathbf{x}} = \mathbf{0}$). In effect, then, the first term of the general equation (3.8) selects a joint velocity vector which produces the desired change in the end-effector position, while the second term exploits the redundancy of the manipulator by varying these joint velocities in such a way that $H(\mathbf{q})$ is minimized without disturbing the end-effector position. By exploiting redundancy in this manner, secondary goals have been created to avoid collisions with obstacles [Bai86], to exploit joint range availability [GM85, KH83], and even to maintain manipulator dexterity by avoiding kinematic singularities [SS87].

3.2.2 Singularities

The pseudoinverse method outlined above provides useful solutions to (3.7) when the Jacobian matrix $\mathbf{J}$ is rectangular, and therefore not invertible. But we must also consider the case where $\mathbf{J}$ is not invertible because it is *singular*. A matrix is said to singular when two or more rows are linearly dependent, and a manipulator is said to be in a *singular configuration* when the Jacobian becomes singular. Figure 3.2 depicts a simple example of a 3-jointed manipulator in a singular configuration. In this example, an incremental change to any of the joint angles will result in approximately the same movement of the end-effector in the y direction; no combination of joint velocities will produce an end-effector velocity in the singular (i.e. x) direction. [1] The Jacobian matrix computed for this configuration will contain zeroes in the first row, and is therefore singular and cannot be inverted.

The pseudoinverse can still be applied to obtain a useful solution when $\mathbf{J}$ is singular. However, as a manipulator passes through a singular configuration there are discontinuities in elements of

[1] Although intuitively it might seem obvious that a rotation at any joint will result in at least some movement in the x direction, recall that equation (3.7) deals with instantaneous quantities.

the computed pseudoinverse due to the change in rank of **J** at the singular configuration [Mac90]. Furthermore, as the manipulator approaches this configuration the pseudoinverse tends to produce large joint velocities. Numerical integration techniques typically do not handle such derivative spikes well. The problem manifests itself as a tendency of the manipulator to oscillate wildly around the singular configuration. So, while the pseudoinverse is able to provide a usable solution *at* a singular configuration, its principal drawback is that it does not provide a continuous, stable solution *around* singularities. While industrial robotic manipulators may be programmed to follow trajectories which explicitly avoid singular configurations for just this reason, this is not really an option for an algorithm to be used for computer animation.

The Singular Value Decomposition

Numerical instabilities near singular configurations are a major problem, which raises the question of whether there is a means of detecting and correcting the problem. Probably the most useful tool for analyzing the Jacobian matrix is the *singular value decomposition* (SVD) [PFTV90]. The SVD theorem states that any matrix can be written as the product of three (non-unique) matrices

$$\mathbf{J} = \mathbf{U}\mathbf{D}\mathbf{V}^T \tag{3.9}$$

The procedure for computing the SVD of a matrix is beyond the scope of this discussion, but is well known and described elsewhere [PFTV90, MK89]. The significance of the SVD lies in the interpretation of each of the three matrices **U**, **D**, and **V**.

For an $m \times n$ matrix **J**, **D** is an $n \times n$ diagonal matrix with non-negative diagonal elements known as *singular values*. If one or more of these diagonal elements is zero, then the original matrix is itself singular. Even better, the ratio of the largest singular value to the smallest one, the *condition number* of the matrix, is a measure of how *ill-conditioned* the matrix **J** is. When the condition number is too large [2], then the matrix is ill-conditioned. It is this ill-conditioning that is responsible for the large joint velocities generated by the pseudoinverse near a singular configuration [Mac90].

The other matrices **U** and **V** are orthonormal bases for the *range* and *null space*, respectively, of **J**. For any zero singular values in **D**, the corresponding columns in **V** form a set of orthogonal vectors which span the space of homogeneous solutions to equation (3.7) (i.e. the set of joint velocities which will not move the end-effector). Likewise, the non-zero singular values have corresponding columns in **U** which span the space of solutions which *will* move the end-effector. We will refer to these basis matrices again when discussing constraints in Chapter 5.

While the SVD provides a means for detecting ill-conditioning in the Jacobian matrix, it does

[2] i.e. its reciprocal approaches machine precision limits

not in itself provide a way for dealing with the ill-conditioning. Nevertheless it is useful as an analytical tool. Klein and Huang [KH83] have used singular value analysis to demonstrate the optimal properties of the Moore-Penrose pseudoinverse. Maciejewski [Mac90] has used the SVD to illustrate the discontinuity that occurs in the pseudoinverse at a singularity, and to develop a strategy for damping the high velocities which occur near singular configurations. But the cost of computing the SVD, $O(n^2 logn)$ for an $n \times n$ matrix, adds significantly to the per-iteration cost of any control algorithm, so it is often not feasible to incorporate it into on-line control schemes. Maciejewski [MK89] does describe a method of incrementally updating the SVD from one iteration to the next which reduces the cost to $O(n^2)$ per iteration, but this requires careful implementation to reduce cumulative errors and the cost is still high enough to deter its use.

3.3 Optimization-Based Methods

A fundamentally different approach to solving the inverse kinematic problem avoids the matrix inversion step altogether. The idea is to cast the basic problem of equation (3.4) as a minimization problem, then apply standard iterative non-linear optimization techniques to obtain a solution.

As an example, consider the problem of positioning the end-effector $\mathbf{x}$ at a goal position $\mathbf{p}$. The distance from the current position $\mathbf{x}(\mathbf{q})$ to the goal position $\mathbf{p}$ serves as an error measurement:

$$E(\mathbf{q}) = (\mathbf{p} - \mathbf{x}(\mathbf{q}))^2 \tag{3.10}$$

By varying the joint angle vector $\mathbf{q}$ the end-effector either moves away from $\mathbf{p}$, increasing the error measure, or towards $\mathbf{p}$, decreasing the error. Clearly the intent is to find a joint vector $\mathbf{q}$ which minimizes the error measure. Limits on the joint ranges of motion provide additional constraints on the individual joint values q_i. Formally, we need to find a vector $\mathbf{q}$ which solves the problem

$$\begin{array}{ll} \text{minimize} & E(\mathbf{q}) \\ \text{subject to} & l_i \leq q_i \leq u_i \quad i = 1 \ldots n \end{array}$$

where l_i and u_i are the lower and upper bounds, respectively, on the value of joint variable q_i. For this example, the error measure E is just the distance formula, but the approach generalizes to more complex goals for the end-effector since $E(\mathbf{q})$ can be any arbitrary function of the joint vector $\mathbf{q}$.

This formulation is a classic non-linear constrained optimization problem, which can be solved by a number of standard numerical methods. A good introduction to the topic of optimization and the issues to consider in selecting a solution method is presented by Press et. al. [PFTV90]. Gill et. al. survey a number of practical optimization techniques [GMW81]. The effectiveness of any particular method is usually determined by the characteristics of the objective function, in this case E, and

of the constraints. For our example, a solver for minimizing smooth quadratic functions subject to linear inequality constraints would be an appropriate choice.

A typical solver will iteratively converge from an initial state towards a solution state, at each step perturbing the state variables slightly, and reevaluating the objective function to evaluate its progress [3]. Some solvers may make use of the gradient of the objective function $\nabla \mathbf{E}$ to suggest new directions in which to perturb the state vector. This may increase the computation per iteration, but pay off in an improved rate of convergence toward a solution.

Once selected, the optimizer can be thought of as a "black box" which is fed as inputs: the current joint vector $\mathbf{q}$, a function for evaluating the objective function E, and possibly a function to evaluate $\nabla \mathbf{E}$ as well. The output from the "black box" is a new joint vector which minimizes the error measure E and therefore solves the inverse kinematic problem.

3.3.1 Evaluation

While conceptually simple, there are some practical difficulties in implementing this approach. Constrained optimization of arbitrary non-linear functions is still an open research area, which has produced a collection of numerical methods which may or may not work for a particular problem. Selecting an appropriate solver, and determining what the problem is when it fails to work, can be difficult. A solver may work well for one particular type of problem, but fail miserably on others.

Furthermore, there is no guarantee that a solver will find the true *global* minimum for a constrained optimization problem. Since most solvers converge on a solution by iteratively moving "downhill" along the objective function surface, they cannot distinguish between a true global minimum and merely a local minimum of the surface. In practical terms, this implies that the solver may return a joint vector $\mathbf{q}$ which does not provide the best solution, and the user may have to somehow suggest a more appropriate configuration from which to restart the iterative search for a better solution.

For interactive computer graphics, the demands of interactivity place some additional demands on the solver. Interactive dragging of a manipulator involves repeatedly sampling the cursor location onscreen to determine a goal position for the end-effector, then invoking the solver with the current manipulator state as the initial guess at a solution. To maintain the illusion of continuous interactive control during dragging, the screen needs to be updated at a reasonable refresh rate [4]. If the solver "black box" cannot produce a solution quickly enough to provide good feedback to a user dragging

[3] each step should decrease (or increase, when maximizing) the objective function

[4] 10 frames/sec, for example, is a minimal goal to aim for during interaction

a manipulator onscreen, then the interface will feel sluggish and unresponsive. A related problem is that optimizers typically work best when the initial state is close to the final solution. But during interactive dragging the cursor may get ahead of the solver, so that on the next invocation of the solver the state of the manipulator being dragged is not close to the next solution. As a result the solver may drastically alter the state while solving for the next solution: the result is that the manipulator will seem to suddenly jump to a completely different configuration in order to satisfy the goal. This is of course quite disconcerting to a user. One option is to interrupt the solver and obtain its current state in order to refresh the screen. However, since the solver is free to try different paths as it "feels" its way towards the closest minimum, there is no guarantee that intermediate solutions will be suitable for refreshing the screen.

3.4 Applications to Computer Graphics

Each of the approaches above have previously been adopted for computer graphics. The pseudoinverse method was introduced to the computer graphics community by Girard in 1985 [GM85] for his *PODA* gait generator. Girard exploited the redundancy of animal limbs in an attempt to minimize joint limit violations, using the projection operator method of Section 3.2.1. The inverse kinematic capabilities of Sims' gait controller [SZ88] and Chadwick's *Critter* system are also based on the technique. None of these efforts suggest specific solutions to the problems the method exhibits around singularities, so it seems reasonable to assume that each of these systems will not perform well near singular configurations.

Badler and Zhao [CP90, ZB89] have adopted the second approach for the *Jack* system, applying a variable-metric optimization procedure to provide interactive control over an articulated figure's posture. Joint range limits are presented as constraints to the optimizer, and a number of objective functions for simple geometric constraints are developed. These include, for example, point-to-point constraints, point-to-plane constraints, orientation constraints and others of a similar nature. Simultaneous constraints on multiple body parts can be imposed, by simply summing the individual objective functions for each constrained part into an aggregate objective function to be minimized. This permits inverse kinematic manipulation of a figure while maintaining a set of constraints on the body, which is a useful interactive tool. Phillips [PB91] even describes an objective function which attempts to balance the *Jack* figure, providing a good example of how the method can be extended to handle arbitrarily complex non-geometric goals. While *Jack's* capabilities are impressive, and do perform at interactive speeds, it works best on high-performance graphics workstations, and even then the imposition of just a few constraints noticeably degrades the response time of the interface. Badler admits to periodically refreshing the screen with intermediate solutions from the optimizer to retain interactivity, even after stating [CP90] that only the final solution is useful and that the

intermediate solutions should not be considered "motion".

A simpler conjugate-gradient minimization technique is also used by Alt and Nicolas [AN88], providing inverse kinematic manipulation and animation of limbs by animating position goals over time. In constrast to Badler's approach they perform unconstrained minimization, electing to enforce joint limits by simply clamping solution values to the joint variable bounding values.

It is also worth noting that some early attempts by Badler to solve the inverse kinematic problem were based on simple heuristic algorithms [KB82, BMW87]. These methods do not appear to have gained wide acceptance; Badler has since abandoned these approaches in favour of the optimization method described above.

Chapter 4

Efficient Algorithms for Direct Manipulation

One of our goals is to provide direct manipulation of an articulated figure, and for this the methods of the previous chapter have some shortcomings. The pseudoinverse is expensive to compute and is subject to numerical instabilities around singular configurations. Badler's optimization method is considerably more powerful, but its performance leaves something to be desired and it may be more suitable for solving problems off-line than for interactive manipulation. The continuing appearance of inverse kinematic papers in the robotics literature seems to suggest that neither of these methods is entirely satisfactory.

In this chapter, a pair of simple inverse kinematic algorithms are presented as alternatives to those adopted for computer graphics in the past. Both are relatively simple to implement, are numerically stable, and are efficient enough to provide reasonable interactive performance on even low-performance machines. Each method exhibits a different behaviour than the other, and it is suggested that they might work well together as complementary manipulation tools.

4.1 A Simplified Dynamic Model

The first method belongs to a class of solutions to the inverse kinematic problem that are based on the transpose of the Jacobian matrix. Like the resolved-motion rate control of section 3.2, it relies on the linear relationship between end-effector and joint velocities. Unlike this other approach, however, no inversion of the Jacobian matrix is required, which significantly reduces the cost of each iteration. Consequently, the method has been advocated for the on-line, dynamic control of robotic manipulators.

The method was introduced in 1984 by Wolovich and Elliot [WE84], who described a dynamic control scheme based on the use of the Jacobian transpose, and showed that it could provide stable tracking of an arbitrary end-effector trajectory. Sciavicco and Siciliano [SS87] applied the method to redundant manipulators, and showed that the redundant degrees of freedom could be used to satisfy both obstacle avoidance constraints, and constraints on joint ranges of motion. Das et. al. [DSS88] develop a more general technique for satisfying secondary criteria, similar to the pseudoinverse-based method discussed in section (3.2.1), and compare the method to a minimization algorithm based on Newton's method. Novakovic and Nemec [NN90] show that the method can be used to generate either joint velocities or joint accelerations. We are interested here in producing joint velocities to drive the joint parameters.

4.1.1 The Jacobian Transpose Method

Consider a composite force $\mathbf{F}$ applied to the tip of a (real) manipulator, consisting of both a pull $\mathbf{f}$ in some direction, and a twist (torque) $\mathbf{m}$ about some axis

$$\mathbf{F} = [f_x, f_y, f_z, m_x, m_y, m_z]^T \tag{4.1}$$

This external force $\mathbf{F}$ applied to the end-effector will result in internal torques and forces at the manipulator joints. Under the simplifying assumption of the *principle of virtual work* [Pau81], the relationship between $\mathbf{F}$ and the vector of internal *generalized* forces τ is

$$\tau = \mathbf{J^T F} \tag{4.2}$$

This suggests a rather simple iterative method for forcing an end-effector to track a time-varying trajectory $\mathbf{x}_d(t)$. If the current end-effector position is given by $\mathbf{x}_c(t)$, then the error measure

$$\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}_c(t) \tag{4.3}$$

can be thought of as a force $\mathbf{f}$ pulling the end-effector toward the desired trajectory point $\mathbf{x}_d(t)$. Equation (4.2) can then be used to transform this external "force" to a generalized force on each of the joint variables. If we are interested in the dynamic behaviour of the manipulator in reaction to the applied force, then τ can be considered the vector of joint variable accelerations $\ddot{\mathbf{q}}$. Since we are not particularly interested in accurate dynamic simulation of the manipulator, it suffices to think of τ as the vector of joint displacements (velocities), so that

$$\dot{\mathbf{q}} = \mathbf{J^T F} \tag{4.4}$$

Once $\dot{\mathbf{q}}$ is computed, a single integration step yields a new vector $\mathbf{q}$ which moves the end-effector towards $\mathbf{x}_d(t)$. This procedure repeats until the end-effector reaches the desired position, or some other stopping criterion is met.

In effect, at each iteration we treat the vector $\mathbf{e}$ as an elastic force pulling on the end of a dynamically simple manipulator, and adopt a heuristic simulation model in which force is proportional to velocity, rather than acceleration. Using $f = mv$ as the governing equation of motion eliminates the effects of inertia; things stop moving as soon as the applied external force disappears. Gleicher [GW91] has used the relationship (4.4), combined with this simple equation of motion, to provide direct manipulation over a variety of geometric objects. The term *differential manipulation* has been coined to describe this, and the discussion above shows that we can add articulated structures to the set of objects amenable to direct manipulation of this type.

Equation (4.4) is in fact equivalent to the well-known *steepest descent* method of minimization, which is generally regarded as having poor convergence properties [PFTV90]. This raises the question of why one would adopt this approach over other minimization methods with superior convergence properties. The intent here is to provide direct manipulation of articulated figures in an interactive setting, rather than to solve inverse kinematic problems off-line, and for this type of on-line application, the Jacobian transpose method has some appealing characteristics.

The first of these is that no matrix inversion is required. Using the transpose of the Jacobian not only avoids problems with matrix singularities, but also means that at each iteration only forward kinematic calculations are required. Since these can be computed quickly, a trajectory which is being specified interactively can be sampled frequently, providing responsive feedback to the user.

More importantly, since the solution is based on a physical model, albeit a simplified one, the solutions obtained in successive iterations are both predictable and intuitive. That is, the manipulator will respond as though the user were in fact pulling on the end-effector with an elastic band. In contrast, other minimization techniques may generate successive solutions which are quite different from each other, particularly when the manipulator is redundant and there are an infinite number of acceptable solutions. When using a physically-based model, successive configurations are implicitly constrained to be close to each other; the "pulling" metaphor uniquely specifies a path from one solution to the next. For interactive applications this advantage arguably outweighs the potentially poor convergence properties of the method.

One drawback is that since the method is based on the Jacobian it is not entirely immune to kinematic singularities, since the matrix will be inherently ill-conditioned. Near a singularity high joint velocities can result in oscillations about the singular configuration. This typically occurs only when trying to fully extend the manipulator. The problem can be overcome by either using a smaller integration stepsize, using an integration method with an adaptive stepsize, or by clamping joint velocities to some maximum bounds before integrating. Since the basic solution step is so quick

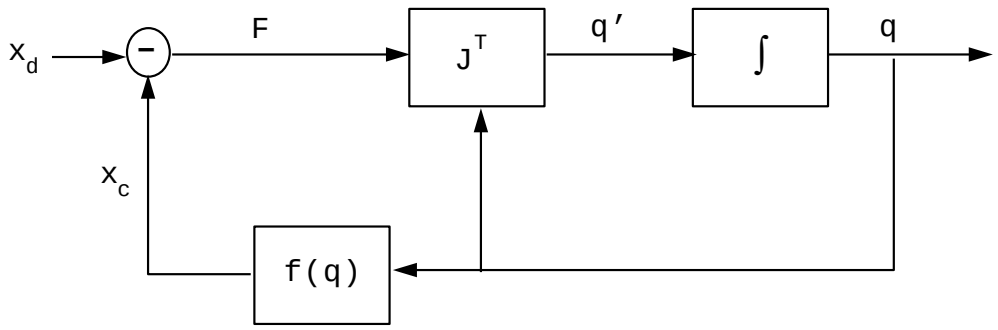


Figure 4.1: Interactive control loop model for Jacobian transpose method

to compute, often it is possible to simply use a smaller integration stepsize while retaining good interactive performance.

4.1.2 Implementation Details

In comparison with the methods of the previous chapter, implementation of the Jacobian transpose method is relatively simple. Each iteration involves computing a force applied to the end-effector, computing the Jacobian matrix for the current configuration, and computing the resulting joint velocities. The joint velocities are integrated once to obtain a new configuration for the manipulator. Figure 4.1 depicts the basic control loop.

Forces applied to the end-effector are a result of interaction with the user, and will be discussed in Chapter 6. For now we assume that the applied force is known, and concentrate on the other aspects of the method.

4.1.3 Computing the Jacobian

At each iteration, we need to compute the Jacobian matrix $\mathbf{J}$ whose columns convey how the end-effector frame moves in the world frame as the individual joint variables change. Given a vector of joint variables $\mathbf{q}$, the end-effector frame is specified by a position $\mathbf{P}(\mathbf{q})$ and orientation $\mathbf{O}(\mathbf{q})$. The

Jacobian column entry for the i'th joint is

$$\mathbf{J}_i = \begin{bmatrix} \partial P_x \\ \partial P_y \\ \partial P_z \\ \partial O_x \\ \partial O_y \\ \partial O_z \end{bmatrix} \tag{4.5}$$

where the derivative operator is with respect to q_i. These entries can either be computed directly in the world frame, or may be computed in the local joint frame before being transformed to the world frame. Here we describe both procedures for efficiently computing the full Jacobian matrix for a manipulator.

Each joint i in the manipulator either translates along or rotates about one of the principal axes $\mathbf{u}_i$ in the local joint frame. At each joint, $\mathbf{M}_i$ is the matrix which transforms the local joint frame to the world frame. Suppose that $\mathbf{axis}_i$ represents the normalized transformation of the local joint axis in the world coordinate frame

$$\mathbf{axis}_i = \mathbf{u}_i \mathbf{M}_i \tag{4.6}$$

Then for a translating joint, the corresponding Jacobian entry is

$$\mathbf{J}_i = \begin{bmatrix} [\mathbf{axis}_i]^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.7}$$

and for a rotating joint the Jacobian entry is

$$\mathbf{J}_i = \begin{bmatrix} [(\mathbf{p} - \mathbf{j}_i) \times \mathbf{axis}_i]^T \\ [\mathbf{axis}_i]^T \end{bmatrix} \tag{4.8}$$

where $\mathbf{p}$ denotes the position of the end-effector, and $\mathbf{j}_i$ is the position of joint i in the world.

Local vs. World Frame

In practice, it is expected that the manipulator itself will be a sub-tree embedded within a general transformation hierarchy within an animation system. This sub-tree may contain arbitrary transformation nodes in addition to those corresponding to the manipulator joints. These additional transformation nodes do not constitute degrees of freedom, and are not subject to updates by inverse kinematic manipulation. But since they transform some of the local joint frames of the manipulator

they must be considered. The significance of this is that the types of node present in the hierarchy determine whether the Jacobian entries can be computed in the global world frame, or whether they must be computed in the local joint frames at a slightly higher cost.

If the transformation hierarchy in which the manipulator is embedded consists only of translations and rotations, then the Jacobian entries can be computed directly in the world frame. A concatenation of only rotations and translations guarantees that the upper-left 3×3 portion of the transformation $\mathbf{M}_i$ at a joint will be orthogonal. This implies that the first three rows of $\mathbf{M}_i$ are the transformed principal axes of the joint's local frame, and so $\mathbf{axis}_i$ can be extracted directly from $\mathbf{M}_i$. Similarly, the joint position $\mathbf{j}_i$ can be extracted from the fourth row of $\mathbf{M}_i$. Since the transformation matrices $\mathbf{M}$ need to be computed to display the structure, caching them at each joint during display reduces the calculation of each Jacobian entry $\mathbf{J}_i$ to just a vector subtraction and a cross product operation.

If scaling transformations are permitted in the hierarchy [1], then there is no guarantee that $\mathbf{M}_i$ is orthogonal. Arbitrary, non-uniform scaling transformations within the manipulator may result in an upper-left 3×3 portion of $\mathbf{M}_i$ which does not represent a pure rotation. In this case, $\mathbf{axis}_i$ cannot be extracted from $\mathbf{M}_i$. Instead the vector quantities in (4.7) and (4.8) must be computed in the local joint frame, then transformed to the world frame by $\mathbf{M}_i$. To compute (4.8) in the local frame, the end-effector position $\mathbf{p}$ needs to be transformed from the world frame to the joint frame, and so the inverse transformation $\mathbf{M}_i{}^{-\mathbf{1}}$ is required.

For efficiency, $\mathbf{M}_i{}^{-\mathbf{1}}$ can be computed incrementally while traversing the hierarchy. Using the matrix identity

$$(\mathbf{AB})^{-\mathbf{1}} = \mathbf{B}^{-\mathbf{1}}\mathbf{A}^{-\mathbf{1}} \tag{4.9}$$

and the fact that the inverse transformation for a single joint is simple to determine, the matrix inversion step at a joint can be replaced with a cheaper matrix multiplication. Further simplifications result from noting that the world frame joint position $\mathbf{j}_i$ corresponds to the origin of the local joint frame, and that the joint axis is simply one of the principal axes, making the cross product step trivial.

The Jacobian matrix for the end-effector frame, then, can be assembled by a single traversal of the hierarchy. At each joint i, either (4.7) or (4.8) is computed to obtain the column $\mathbf{J}_i$, and these calculations may be performed either in the global frame or the local joint frame.

[1] as they are likely to be in any computer animation system

4.1.4 Scaling Considerations

In practice, the simple model expressed in equation (4.4) does not work well for the general case. One problem is that the response of the manipulator to an applied force depends on the choice of scale for the joint variables. If rotation joint values represent degrees, for example, a given applied force will result in a different response than if the rotations were measured in radians. The overall scale of the world must also be considered, since this will govern the magnitude of the vector quantities in (4.4). These factors must be taken into account if the technique is to provide uniform behaviour over a range of scales.

To provide stable scale-invariant behaviour, we can modify equation (4.4) to compensate for these factors

$$\dot{\mathbf{q}} = \mathbf{K}\mathbf{J}^{\mathbf{T}}\mathbf{F} \tag{4.10}$$

where $\mathbf{K}$ is simply a constant scaling matrix whose i'th diagonal entry $\mathbf{K}_i$ acts as a weighting factor for the computed joint velocity $\mathbf{q}_i$. The following term is suggested

$$\mathbf{K}_i = \frac{1}{w_i \alpha_i} \tag{4.11}$$

where the w_i term is proportional to the length of link i and is intended to offset the effects of the overall scale of the world. The α_i term is a weighting factor for joint i, which can be thought of as a parameter controlling the responsiveness of the joint to an applied force. This parameter might be made available to the user to allow editing of a joint's perceived "stiffness".

Although $\mathbf{K}$ is specified here to be a constant gain matrix, it is possible to compute a time-varying gain matrix $\mathbf{K}(t)$ so as to control the rate of convergence of a tracking algorithm based on (4.10) [WE84, NN90]. However, given that the additional computation required is considerable relative to the basic iteration cost of (4.10), there is some incentive to just consider $\mathbf{K}$ constant.

4.1.5 Integration

Once the joint velocities $\dot{\mathbf{q}}$ are known, a single integration step is required to generate the new state of the manipulator, $\mathbf{q}$, for the next iteration. The simplest method is to take a single Euler integration step

$$\mathbf{q} = \mathbf{q} + h\dot{\mathbf{q}} \tag{4.12}$$

for some integration stepsize h. This method is notoriously inaccurate, since it assumes constant velocity across the step interval. As the interval width h is made smaller the accuracy improves, but overall performance suffers. Nevertheless, the Euler method with a small h may be adequate for manipulators with just a few joints.

The Euler method can be replaced with a more robust method. An adaptive stepsize Runge-Kutta method can improve performance, taking small steps when required but allowing the stepsize h to increase when small steps are unnecessary. Adaptive integration implementations are available in the public domain [PFTV90], and their use is recommended.

4.1.6 Joint Limits

Since the formulation does not explicitly include constraints on the joint variable values, joint limits are enforced by clamping q_i to upper and lower bounds after the integration step. Although this is not recommended in most optimization texts, in practice it appears to be adequate.

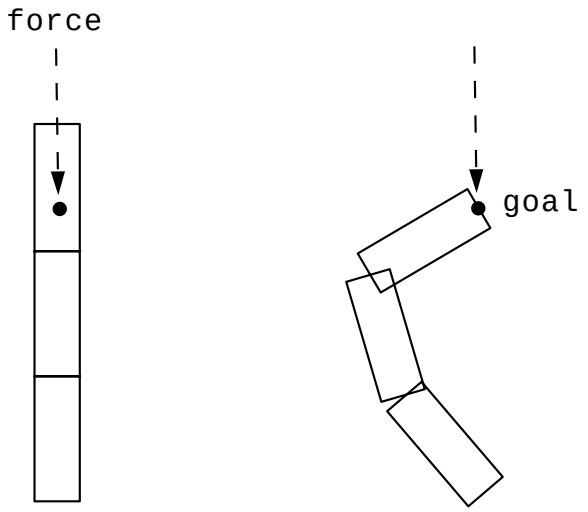


Figure 4.2: A case not handled well with the Jacobian transpose method. Pulling inwards on the tip of the manipulator on the left will not produce an expected configuration like the one shown on the right.

4.2 A Complementary Heuristic Approach

The force-based approach of the preceding method has some appealing characteristics for interactive manipulation. There are some cases, however, for which the method does not perform well. Figure 4.2 illustrates one such case. On the left is a manipulator in a singular configuration, with a new, desired position for the tip shown as a black dot. The configuration on the right shows a reasonable solution to this inverse kinematic problem, and probably reflects what a user expects to get by dragging the tip in towards the goal. However, as the user attempts to drag the tip inwards the applied force exerted on the tip points straight down the singular direction; it is in effect cancelled

out, and the tip will not move. This behaviour is reasonable, given the physical analogy of the Jacobian transpose method, but may not really match the user's expectation.

An alternative inverse kinematic algorithm is presented here, suitable for interactive positioning, and capable of providing reasonable behaviour in cases such as that in Figure 4.2. It is based on a heuristic method which has been proposed to quickly find an initial feasible solution for a standard minimization-based algorithm [WC91]. However it can stand on its own as an inverse kinematic positioning tool. Like the Jacobian transpose method, the technique is efficient, simple, and immune to problems with singularities. However its behaviour is quite different from that exhibited by the previous algorithm, and it is suggested here as a complement to, rather than a replacement for, the Jacobian transpose method.

4.2.1 The Cyclic-Coordinate Descent Method

The *cyclic-coordinate descent* (**CCD**) method is an iterative heuristic search technique which attempts to minimize position and orientation errors by varying one joint variable at a time. Each iteration involves a single traversal of the manipulator from the most distal link inward towards the manipulator base. Each joint variable q_i is modified in turn to minimize an objective function. The minimization problem at each joint is simple enough that an analytic solution can be formulated, so each iteration can be performed quickly.

As a solution is obtained at each joint the end-effector position and orientation are updated immediately to reflect the change. Thus the minimization problem to be solved at any particular joint incorporates the changes made to more distal joints during the current iteration. This differs from the previously described method, which in effect determines the changes to each joint simultaneously [2].

Suppose that the current end-effector position is

$$\mathbf{P}_c = (x_c, y_c, z_c)$$

and that the current orientation of the end-effector is specified by the three orthonormal rows of the rotation matrix $\mathbf{O}_c$

$$\mathbf{O}_c = \begin{bmatrix} \mathbf{u}_{1c} \\ \mathbf{u}_{2c} \\ \mathbf{u}_{3c} \end{bmatrix}$$

[2] i.e. although the changes are computed sequentially, the state of the manipulator remains constant during the iteration

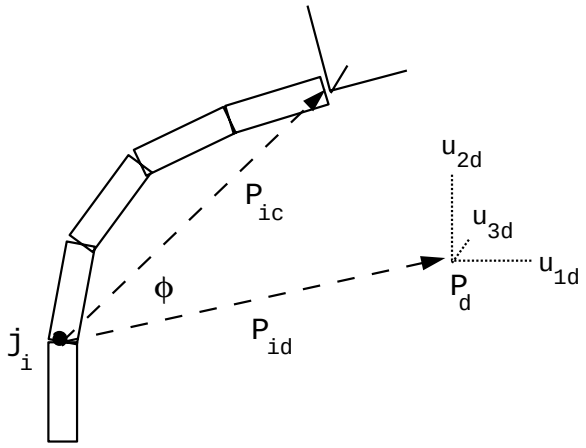


Figure 4.3: Example CCD iteration step for rotation joint i.

The end-effector can be placed as close as possible to some desired position $\mathbf{P}_d$ and orientation $\mathbf{O}_d$ by finding a joint vector $\mathbf{q}$ which minimizes the error measure

$$E(\mathbf{q}) = E_p(\mathbf{q}) + E_o(\mathbf{q}) \tag{4.13}$$

which is just the sum of a positional error measure

$$E_p(\mathbf{q}) = \|(\mathbf{P}_d - \mathbf{P}_c)\|^2 \tag{4.14}$$

and an orientation error measure

$$E_o(\mathbf{q}) = \sum_{j=1}^{3} ((\mathbf{u}_{jd} \cdot \mathbf{u}_{jc}) - 1)^2 \tag{4.15}$$

The method proceeds by considering one joint at a time, from the tip to the base. Each joint variable q_i is modified to minimize equation (4.13), before proceeding to the next joint $i-1$. At each joint, minimizing (4.13) becomes a simple one-dimensional optimization problem, since only q_i is allowed to change while the other elements of $\mathbf{q}$ are fixed. Since joint i is either a rotation or a translation joint, there are two cases to be considered.

Rotation Joint

Figure 4.3 illustrates the situation for rotation joint i during an iteration. The vector $\mathbf{P}_{ic}$ is the vector from the joint position $\mathbf{j}_i$ to the current end-effector position, and $\mathbf{P}_{id}$ is the vector from $\mathbf{j}_i$ to the desired end-effector position. We are free to rotate the vector $\mathbf{P}_{ic}$ about the world space joint axis $\mathbf{axis}_i$ by some amount ϕ. This rotated vector is

$$\mathbf{P}'_{ic}(\phi) = \mathbf{R}_{\mathbf{axis}_i}(\phi)\mathbf{P}_{ic}$$

As ϕ varies, $\mathbf{P}'_{ic}(\phi)$ sweeps out a circle centered at $\mathbf{j}_i$. The point on this circle closest to the desired position $\mathbf{P}_d$ is the point at which the circle would intersect the line along $\mathbf{P}_{id}$, so the best we can do by varying joint variable j_i alone is to align the two vectors $\mathbf{P}'_{ic}(\phi)$ and $\mathbf{P}_{id}$. This implies that we seek a value for ϕ which *maximizes* the expression

$$g_1(\phi) = \mathbf{P}_{id} \cdot \mathbf{P}'_{ic}(\phi) \tag{4.16}$$

Reasoning along similar lines, an orientation error is best corrected by making sure that ϕ also *maximizes* the expression

$$g_2(\phi) = \sum_{j=1}^{3} \mathbf{u}_{j\,d} \cdot \mathbf{u}'_{j\,c}(\phi) \tag{4.17}$$

Combining both (4.16) and (4.17) gives an aggregate objective function to be maximized for joint i

$$g(\phi) = w_p g_1(\phi) + w_o g_2(\phi) \tag{4.18}$$

Here w_p and w_o are arbitrary position and orientation weighting factors, respectively, and are introduced to play a role similar to that of the gain matrix $\mathbf{K}$ of the Jacobian transpose method. The following *ad hoc* values for these factors are suggested [WC91]

$$\begin{aligned} w_o &= 1 \\ w_p &= \alpha(1+\rho) \end{aligned}$$

Here α is a scaling factor inversely proportional to the overall world scale W

$$\alpha = k/W$$

and is required to make the algorithm's behaviour scale-invariant. The factor ρ is an arbitrary weight which depends on the configuration of the manipulator; Wang [WC91] suggests, without justification, that

$$\rho = \frac{\min(\|\mathbf{P}_{id}\|, \|\mathbf{P}_{ic}\|)}{\max(\|\mathbf{P}_{id}\|, \|\mathbf{P}_{ic}\|)}$$

which seems to be adequate in practice.

With some algebraic manipulation, the objective function (4.18) to be maximized at joint i can be reduced to

$$g(\phi) = k_1(1 - cos\phi) + k_2 cos\phi + k_3 sin\phi \tag{4.19}$$

with the constant coefficients k_1, k_2, and k_3 given by

$$k_1 = w_p(\mathbf{P}_{id} \cdot \mathbf{axis}_i)(\mathbf{P}_{ic} \cdot \mathbf{axis}_i) + w_o \sum_{j=1}^{3} (\mathbf{u}_{j\,d} \cdot \mathbf{axis}_i)(\mathbf{u}_{j\,c} \cdot \mathbf{axis}_i) \tag{4.20}$$

$$k_2 = w_p(\mathbf{P}_{id} \cdot \mathbf{P}_{ic}) + w_o \sum_{j=1}^{3} (\mathbf{u}_{jd} \cdot \mathbf{u}_{jc}) \tag{4.21}$$

$$k_3 = \mathbf{axis}_i \cdot \left[w_p(\mathbf{P}_{ic} \times \mathbf{P}_{id}) + w_o \sum_{j=1}^{3} (\mathbf{u}_{jc} \times \mathbf{u}_{jd}) \right] \tag{4.22}$$

From elementary calculus, we know that the objective function (4.19) is maximized over the interval $-\pi \leq \phi \leq \pi$ when its first derivative is zero and its second derivative is negative. The first condition

$$(k_1 - k_2)sin\phi + k_3 cos\phi = 0$$

implies that

$$\phi = tan^{-1} \frac{k_3}{(k_2 - k_1)} \tag{4.23}$$

which determines a candidate value ϕ_c in the range $\frac{-\pi}{2} < \phi_c < \frac{\pi}{2}$. However, since tan is periodic there are potentially two other candidate values to consider: $\phi_c + \pi$ and $\phi_c - \pi$. Of these candidate values, those which lie in the interval $-\pi < x < \pi$ and which pass the second derivative test are maximizing values for the objective function (4.19). If there is more than one of these, the objective function is evaluated with each to determine which yields the true maximum. Once ϕ has been uniquely determined in this way it is added to the current joint value q_i. At this point we can introduce an arbitrary weighting factor w_i, $0 \leq w_i \leq 1$ which controls the perceived "stiffness" of the joint, so that the update becomes

$$q_i = q_i + w_i \phi$$

The end-effector frame is then rotated to reflect this change, and the iteration continues on to the next joint $i - 1$ using the updated end-effector.

Translation Joint

If joint i is a translation joint, then it can only reduce the position error (4.14). It is not difficult to show [WC91] that the best that can be done to minimize the position error is to change the joint displacement by

$$\lambda = (\mathbf{P}_{id} - \mathbf{P}_{ic}) \cdot \mathbf{axis}_i \tag{4.24}$$

This is weighted by w_i, as before, and added to the current value of joint variable q_i. The end-effector position is updated before continuing on to the next joint.

4.2.2 Overview

A single iteration of the CCD method for an n-jointed manipulator visits joints n through 1 in turn. At each joint, the original n-dimensional optimization problem is reduced to a one-dimensional

problem involving just the joint variable q_i, which admits to an analytic solution. An incremental change to q_i is computed with either (4.23), if the joint rotates, or with (4.24) if the joint translates. The variable q_i is then incremented and clamped to upper and lower bounds. The current end-effector frame ($\mathbf{P}_c$ and $\mathbf{O}_c$) is updated to reflect the change before proceeding to the next joint.

The algorithm behaves well around singular configurations, and since the value of the objective function (4.13) is reduced with each step, the method is guaranteed to converge. But the heuristic nature of the method makes the rate of convergence somewhat difficult to quantify, since it is dependent on the structure of the manipulator itself. In practice, most problems can be solved with only a few iterations, although there are situations for which the method can converge very slowly. In terms of behaviour, the heuristic implies that distal links move more readily than links closer to the base; if the end-effector goal can be reached by moving only the final link of the chain, then only that link will move.

4.3 Comparison

Each of the methods described above is suitable for interactive direct manipulation, with some caveats. The per-iteration cost for each is minimal, so that both can provide good feedback when dragging reasonably complex manipulators. Each can be made numerically stable near kinematic singularities, although the CCD method has an edge in this regard since it is completely immune to difficulties near singularities. Where neither method particularly excels is in the rate at which they converge toward a solution: both can exhibit poor convergence rates, particularly if high accuracy is required.

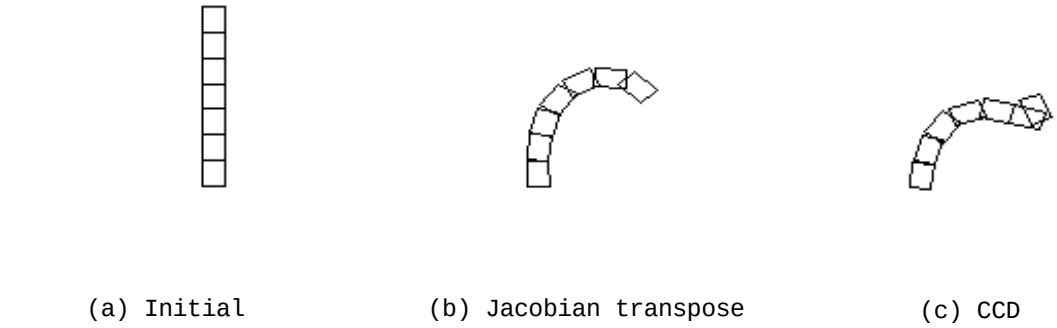
Figures 4.4 and 4.5 compare the performance of each algorithm for solving a pair of inverse kinematic problems. The manipulator has 7 degrees of freedom, about the same complexity as a simple approximation to a limb, although for this example there are no limits on the range of movement for each joint. In each case a position goal is specified for the end-effector, and the inverse kinematic problem is solved to varying degrees of accuracy. Each figure shows the initial configuration, and the final solution obtained with each method. In addition, the time to achieve the solution is plotted with respect to the degree of accuracy requested. In the first case of Figure 4.4 the end-effector goal is well within reach, and both methods are able to solve the problem reasonably quickly. However, it is already apparant that the Jacobian tranpose converges slowly when it is close to a solution; there is a marked decrease in performance for each additional digit of accuracy requested.

In the second case of Figure 4.5, the goal is close to the edge of the reach space of the limb,

which must approach a singular configuration to achieve the goal. Although both methods are able to solve the goal, the CCD method clearly outperforms the other, particularly as the accuracy of the solution increases. As it approaches the goal, the Jacobian transpose method is hampered by a small applied force which is "pulling" in a direction which doesn't affect most of the joints, so the progress towards the goal is very slow. The heuristic approach of the CCD fares much better in this example.

The final configurations shown in the figures also illustrate the CCD method's preference for moving distal links, in contrast with the other method's tendency to distribute joint changes more equally along the chain. This difference in behaviour is quite noticeable during interactive dragging. Manipulating a chain with the Jacobian transpose method tends to feel like playing a flexible elastic rod, while the CCD method imparts a feel more akin to pulling on a chain of loosely connected links. While the final solutions for the CCD method might look inferior to those obtained with the Jacobian tranpose, keep in mind that these solutions were obtained non-interactively - i.e. they were computed off-line. When dragging interactively using the CCD method it is usually not difficult to gain control over the final position by moving the cursor appropriately. Also, both methods support a means of controlling joint responsiveness by specifying an appropriate weighting factor w_i at each joint. In particular, this parameter can be useful to offset the default behaviour of the CCD method.

These examples illustrate the main advantages and disadvantages of each method. Both are quick and therefore worth considering if direct manipulation is required, even on machines with modest performance. The Jacobian method has the advantage of responding in an intuitive fashion to pushes and pulls on the end-effector; the CCD method is not as intuitive in this regard. The CCD method exhibits more stability around singular configurations, and although its rate of convergence slows, it is not nearly to the extent that the Jacobian's does. Moreover, it can be argued that direct manipulation does not necessarily require a high degree of accuracy. Certainly while a user is sweeping a cursor across the screen it is not critical that the end-effector track it to six decimal places of precision; one or two decimal places of accuracy probably suffices. When accurate positioning is required, particularly near singularities, the CCD method would be the appropriate method to use. In general, the two methods complement each other nicely, providing alternate interaction models to offer the user.



(a) Initial (b) Jacobian transpose (c) CCD

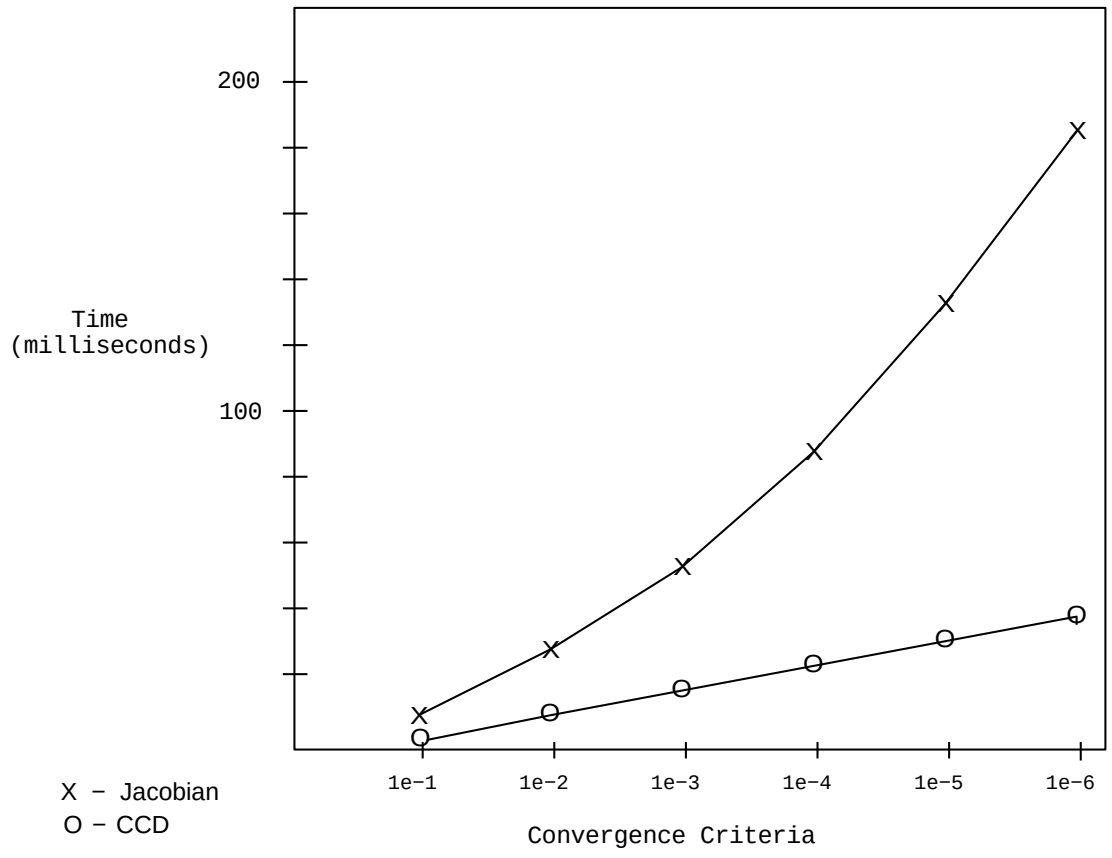


Figure 4.4:

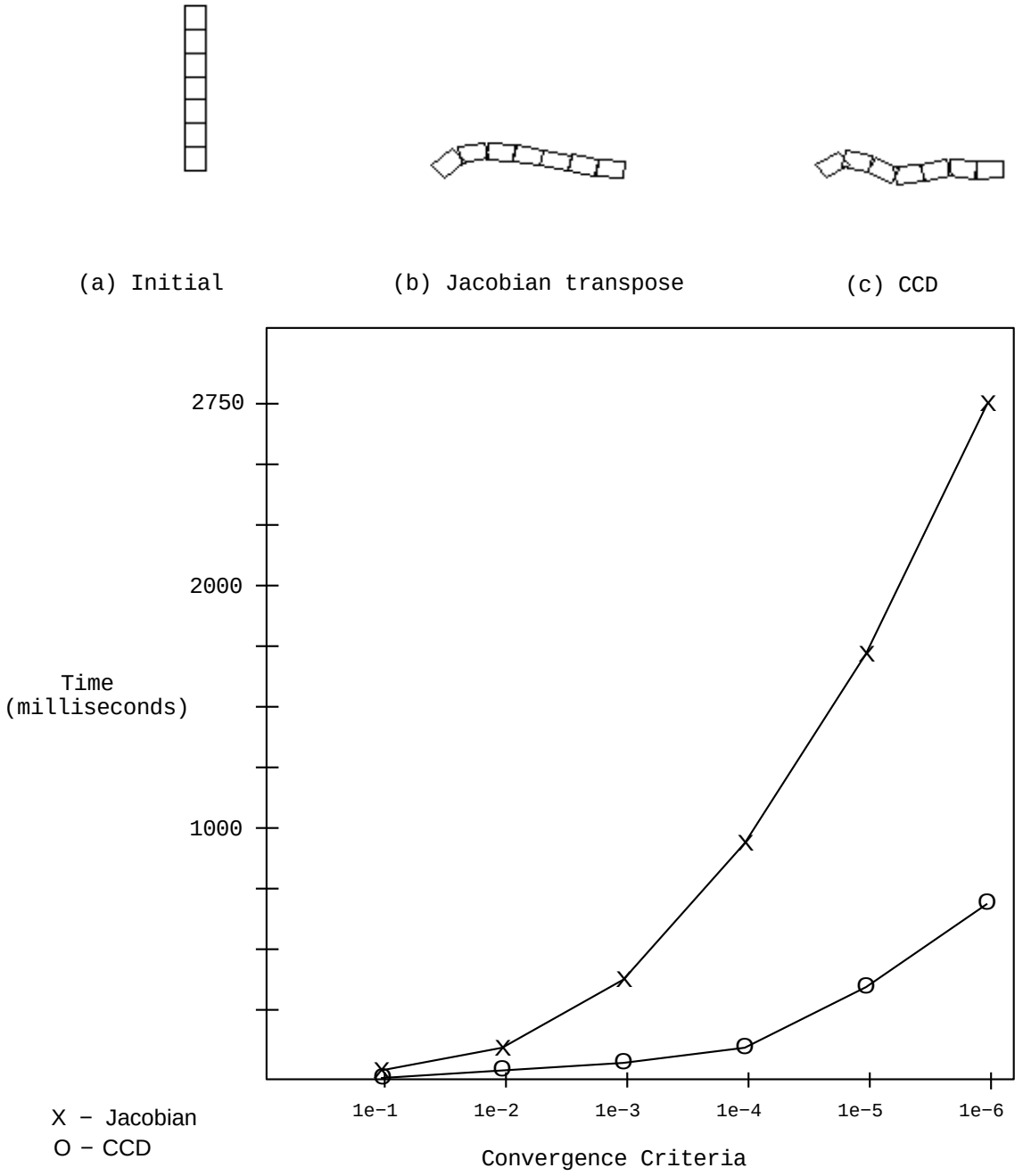

(a) Initial
(b) Jacobian transpose
(c) CCD
2750
2000
1000
Time
(milliseconds)
X − Jacobian
O − CCD
1e−1
1e−2
1e−3
1e−4
1e−5
1e−6
Convergence Criteria

Figure 4.5:

Chapter 5

Incorporating Constraints

Inverse kinematic manipulation is a useful shortcut, but does not in itself provide any more functionality than simple forward kinematics. Creating a pose for a skeleton can be less tedious, but there are still problems in trying to edit the pose without moving previously positioned body parts. It would be useful, for example, to be able to specify that parts of the body should not move, no matter how we manipulate the skeleton. Badler's recent work [CP90] has shown the usefulness of being able to impose constraints on a figure during editing. The constrained optimization method employed with *Jack* permits any constraint which can be expressed as a function of the skeleton state to be specified. However, with the exception of the experimental balance constraint described by Phillips [PB91], this capability has not been exploited. The basic set of constraints available within *Jack* are almost all simple geometric constraints on positions and orientations of body parts. It would seem that even this limited set of constraints is enough to greatly improve interaction with a figure.

In this chapter we consider how each of the manipulation techniques described in the previous chapter can be used to satisfy multiple simple geometric constraints on the positions and orientations of end-effectors within a skeleton, and discuss some of the performance and implementation issues that arise.

5.1 Constraint Satisfaction

A manipulator, or a skeleton, can be considered a system described by a set of state variables - for our purposes, joint rotations. Solving the inverse kinematic problem is essentially one of solving a non-linear system of equations for these state variables, and there are a number of approaches one can take to incorporate constraints in the solution process. We have already seen that constrained optimization is one possibility, but there are other possibilities for the manipulation methods of the previous chapter.

The simplest approach is to use some variant of the *penalty method*. This method can be used with any iterative technique, and the essential idea is that when a constraint is violated a restoring force is introduced to push the system back into a legal state in which all constraints are satisfied. A feedback control loop monitors the state of the system and applies the appropriate penalty forces as it detects constraint violations. An advantage of this approach is that if the constraints are not initially met, the restoring forces pull the system towards a legal state. A disadvantage is that the constraints cannot be enforced exactly, since the restoring force only comes into play *after* a constraint has already been violated. Thus two points constrained to be coincident will appear to pull apart slightly before the constraining force can pull them back together. For methods based on physical simulation, the restoring forces are usually spring-based. If the springs are made sufficiently stiff, then the constraint violations may be small enough to be unnoticeable. But this causes problems when trying to simulate the system, requiring tiny integration steps to retain numerical stability [PFTV90].

Both the Jacobian transpose method and the CCD method could be used within a control loop to provide this sort of constraint satisfaction. In fact, due to the heuristic nature of the CCD method and its reliance on strictly geometric information, this is the only option open. Before describing a penalty-method approach based on the CCD method, we consider first a more comprehensive solution which is a generalization of the Jacobian transpose method.

5.2 Maintaining Constraints

The approach taken is to consider the problem of *maintaining* a set of constraints which are already satisfied, rather than trying to satisfy constraints that have been violated. By assuming that the system is already in a legal state, the problem is reduced to one of ensuring that changes to the system never violate the constraints. The technique is based on well-known methods for implementing constrained dynamics within physical simulations [Sur92, AW90], using the same sort of simplified first-order equations of motion described in Section 4.1.1 and by Gleicher [GW91]. The essential idea is that geometric constraints on an object are treated as mechanical connections which introduce constraining forces into the system being simulated. When an external force is applied to the system, some components of the applied force may be counteracted by these constraining forces, so that the net force acting on the system does not violate the constraints. The nub of the problem is to determine what these constraining forces are, given a set of geometric constraints which must be satisfied.

Figure 5.1 illustrates the concept for the simplest possible system, a point. The system's state $\mathbf{q}$ is simply the point's location in space. A single constraint is imposed on the system, stating that the

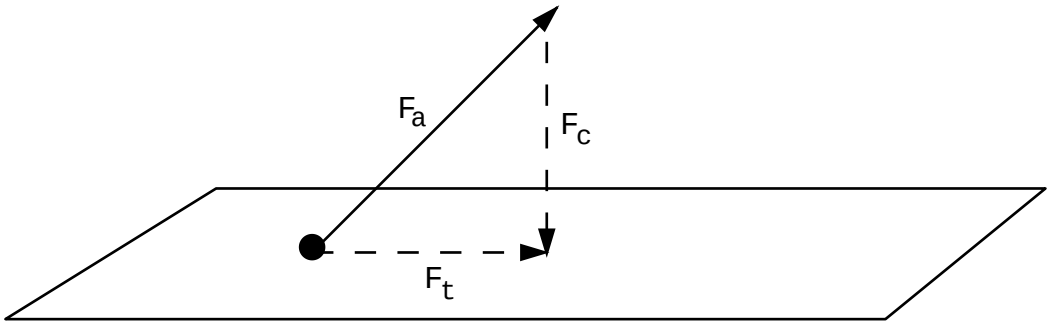


Figure 5.1: A force applied to a point constrained to lie within a plane. A constraining force normal to the plane is added to the applied force to obtain a legal force tangential to the plane.

point must lie within a plane. This constraint condition can be written as a function of the system state, $c(\mathbf{q}) = 0$. Now suppose a force $\mathbf{F_a}$ is applied to the point. If the point is to remain in the plane, then the constraint must introduce a counteracting force $\mathbf{F_c}$ which removes from the applied force those components which would move the point away from the plane. Summing the applied force with this constraint force yields a net force acting on the point which ensures that the point only moves in a "legal" direction. Although simple, this example illustrates two important points which apply to more complicated systems: the net force lies in a plane tangent to the constraint surface $c(\mathbf{q}) = 0$ [1], while the constraining force points in a direction normal to this tangent plane. We now describe a procedure for computing the appropriate constraint forces for arbitrarily complex systems.

5.2.1 The Constraint Condition

Suppose we write the constraints on a system as a vector function of the system state

$$\mathbf{C} = f(\mathbf{q})$$

If the constraints are initially satisfied, then $\mathbf{C} = \mathbf{0}$. In order to maintain the constraints, $\mathbf{C}$ must not change, so the derivatives of $\mathbf{C}$ must always be zero. In particular, this imposes the condition

$$\dot{\mathbf{C}} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{0} \tag{5.1}$$

In Chapter 4, equation (4.10) defined the simplified equations of motion for a system, which had the form

$$\dot{\mathbf{q}} = \mathbf{K}\mathbf{g} \tag{5.2}$$

[1] in fact for this trivial example the tangent plane *is* the constraint surface

for some generalized force $\mathbf{g}$ acting on the system state. Substituting this equation into (5.1), and denoting the constraint Jacobian matrix by $\mathbf{J_C} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}}$, the constraint condition becomes

$$\dot{\mathbf{C}} = \mathbf{J_C K g} = 0 \tag{5.3}$$

Any generalized force $\mathbf{g}$ acting on the system which satisfies this condition will not violate the constraints.

5.2.2 Computing the Constraint Jacobian Matrix

The constraint Jacobian matrix $\mathbf{J_C}$ describes how the individual constraints change as the state variables vary. The section above introduced the constraint vector $\mathbf{C(q)}$, which is made up of m scalar constraint functions $c_1(\mathbf{q}), \cdots, c_m(\mathbf{q})$. If there are n state variables $q_1, \cdots, q_n$, then $\mathbf{J_C}$ is an $m \times n$ matrix. Typically there are more state variables than there are constraints .

In practice, constraints are not specified on $\mathbf{q}$ directly, but rather on more meaningful geometric entities (such as points or orientations), which are themselves functions of the state $\mathbf{q}$. For example, suppose we wish to impose a constraint that the tip of a limb on an articulated skeleton, denoted $\mathbf{P(q)}$, should remain fixed at some location $\mathbf{R}$. An appropriate constraint condition for this example would be

$$(\mathbf{R} - \mathbf{P(q)})^2 = \mathbf{0}$$

In this case the constraint vector $\mathbf{C}$ consists of 3 scalar constraints, one on each of the x, y, and z components of $\mathbf{P}$

$$\begin{aligned} \mathbf{C(q)} &= [\ \ c_1(P_x(\mathbf{q})),\ \ c_2(P_y(\mathbf{q})),\ \ c_3(P_z(\mathbf{q}))\ \ ] \\ &= [\ (R_x - P_x)^2, (R_y - P_y)^2, (R_z - P_z)^2\ ] \end{aligned}$$

Each scalar constraint c_i contributes a single row $\frac{\partial c_i}{\partial q_j}$ to the constraint Jacobian matrix. Thus each geometric constraint contributes a block of r rows to the matrix, where r is the dimension of the geometric quantity (e.g. $r = 3$ for a point constraint). Since geometric constraints are expressed in terms of functions of $\mathbf{q}$, the chain rule is applied to calculate the Jacobian entries. In the case above, for example,

$$\frac{\partial c_i}{\partial q_j} = \frac{\partial c_i}{\partial P_k} \frac{\partial P_k}{\partial q_j}$$

Typically a single constraint will depend on only a few of the system state variables. If the left foot of the sample skeleton of Figure 6.1 is constrained to some location, for example, the arm

joint variables have no effect on this constraint. Thus the i'th row of $\mathbf{J_C}$, corresponding to the constraint c_i, usually contains just a few non-zero elements. It is important that this sparsity in $\mathbf{J_C}$ be exploited and preserved; sparse matrix techniques offer substantial performance gains over a naive implementation.

5.2.3 Computing the Constraint Force

In Chapter 4 it was shown that an external force applied to a point on an articulated skeleton could be converted to a generalized force $\mathbf{g}$ on the skeleton state by the Jacobian transpose method. However, the $\mathbf{g}$ of equation (5.3) is the *net* generalized force on the system. As such, it may include the sum of multiple forces applied to various points on a skeleton. More to the point, it must also include some (unknown) constraining forces which prevent the applied forces from violating the constraints imposed on the system. In general, the net force is considered to be a sum of known applied forces and of unknown constraint forces, $\mathbf{g} = \mathbf{g_a} + \mathbf{g_c}$. Substituting this into the constraint condition (5.3) yields the linear system of equations

$$\mathbf{J_C K g_c} = -\mathbf{J_C K g_a} \tag{5.4}$$

in which only the constraining generalized force vector $\mathbf{g_c}$ is unknown.

Equation (5.4) simply states that an appropriate constraining force is one which, when added to the applied forces, causes the constraint derivatives to be zero. But this is too ambiguous, since the system is usually under-constrained and there may be many constraining forces $\mathbf{g_c}$ which satisfy equation (5.4). One approach to removing this ambiguity is to insist that the constraint force must lie in a direction in which the system may not move [2] (just as the constraining force of Figure 5.1 does). Recalling the discussion in Chapter 3 about the Singular Value Decomposition (SVD), this implies that the constraint force must lie within the *range* of the constraint Jacobian matrix $\mathbf{J_C}$. Thus for constrained systems in general, the constraint force $\mathbf{g_c}$ is restricted to lie in the range of $\mathbf{J_C}$, while the net force $\mathbf{g}$ must lie in its null space. This is a generalization of the specific example presented in Figure 5.1.

If the constraint force is in the range of $\mathbf{J_C}$, then $\mathbf{g_c} = \lambda \mathbf{J_C}$ for some vector λ. Therefore equation (5.4) can be written as

$$\mathbf{J_C K J_C}^T \lambda = -\mathbf{J_C K g_a} \tag{5.5}$$

and solved for the vector of *Lagrange multipliers* λ. Once the Lagrange multipliers are known, the constraint force which removes the illegal components of the applied force is computed as

$$\mathbf{g_c} = \lambda \mathbf{J_C} \tag{5.6}$$

[2] this restriction follows from the *principle of virtual work* [Pau81]

Note that computing $\mathbf{g_c}$ requires the solution to a *linear* system of equations, even though the constraints themselves may be non-linear.

5.2.4 Solving for Lagrange Multipliers

The linear system (5.5) must be solved to find the Lagrange multiplier vector

$$\lambda = [\lambda_1, \cdots, \lambda_m]$$

Note that the number of equations in the system is equal to the dimension of the constraint vector $\mathbf{C}$. Any method for solving sets of linear equations can be applied here.

A method recommended here to solve the system emphasizes robustness over speed. Due to the structure of most articulated skeletons, it is impossible to avoid ill-conditioning in (5.5) all of the time [Mac90]. To cope with this, the system is solved using the *truncated* SVD of the left-hand $m \times m$ matrix $\mathbf{J_c K J_c}^T$, combined with a backsubstition algorithm [PFTV90]. The truncated SVD is formed from the original SVD by zeroing any "small" singular values, which effectively throws away any ill-conditioned components of the matrix [PFTV90, MK89]. This solution method does not take advantage of any sparsity in the matrix, but has the advantage of being robust in cases where ill-conditioning occurs.

5.2.5 Feedback

The discussion so far has been based on the assumption that the constraints are initially satisfied, and that they remain so. To handle cases in which the initial state violates some constraints, or where numerical inaccuracies cause constraint violations, an additional feedback term is added to the basic equation of motion (5.2)

$$\dot{\mathbf{q}} = \mathbf{K}(\mathbf{g_a} + \mathbf{g_c}) - k\mathbf{C}\mathbf{J_c}^T \tag{5.7}$$

This term effectively adds a penalty spring to the system, the strength of which is proportional to the magnitude of the constraint deviation $\mathbf{C}$.

5.2.6 Overview

Equation (5.7) is the complete simplified equation of motion for a constrained system. Constraints are maintained by evaluating the right hand side of the equation, using the methods outlined in the sections above, to find the state variable velocities $\dot{\mathbf{q}}$. A single integration step then yields a new state $\mathbf{q}$ which respects any constraints imposed on the system. In an interactive setting, the procedure iterates as the user applies changes to the system state through the user interface. Screen updates occur after each iteration, to provide feedback. Figure 5.2 provides an overview of the procedure

for a single iteration. Note that equation (5.7) is a generalization of the Jacobian transpose method of chapter 4; in the absence of any constraints on the system, the equation reduces to the simple formula introduced earlier.

5.3 Implementation Issues

The basic constrained equation of motion (5.7) has a regular structure which lends itself to an object-oriented implementation. A solver for the equation only needs to know a few things: the length and value of a global state vector $\mathbf{q}$, how to evaluate a number of functions of $\mathbf{q}$, and how to evaluate a matrix of partial derivatives with respect to $\mathbf{q}$ for each of these functions.

Using software "objects" which respond to specific requests to provide these pieces of information, a generic constraint solver can be written which is insulated from the specific details of articulated skeletons. The solver queries these objects to obtain the information it needs to assemble a global system of constrained equations of motion, solves the system, and then communicates the results back to some of the "objects". Implementing this method of maintaining constraints requires three different types of these "objects": *skeletons*, *handles* and *constraints*.

5.3.1 Skeletons as Objects

Each skeleton is considered a single "object", which contributes variables to the solver's global state vector $\mathbf{q}$. Each skeleton must be able to respond to requests for specific pieces of information that the solver requires. In addition, a method must be provided by which the solver can communicate a solution back to the skeleton. If each skeleton provides these capabilities, the solver does not need to know anything about the internal structure of the skeleton itself. The solver must be able to

- query a skeleton for the length of its state vector
- query a skeleton for its state variable vector
- query a skeleton for its scaling matrix $\mathbf{K}$
- send a skeleton a new state vector reflecting the solution

5.3.2 Handles on Skeletons

A *handle* is an abstraction representing a geometric quantity associated with a skeleton, whose value depends on the skeleton's internal state. A handle can refer to something as simple as the position of a joint within the skeleton, or as complex as the location of the skeleton's center of mass. Handles are the geometric entities upon which constraints may later be imposed, and to which forces may be applied.

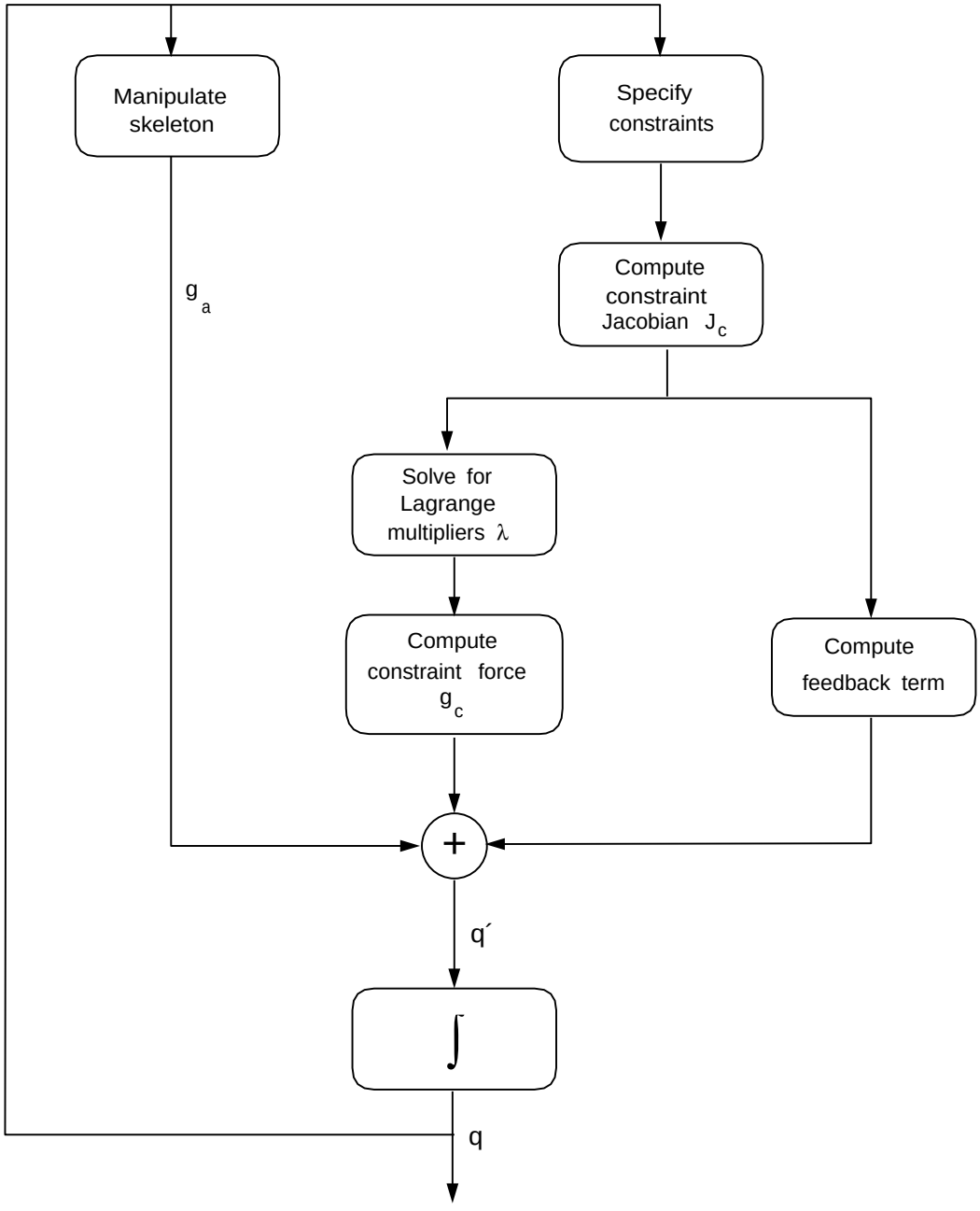


Figure 5.2: Iteration steps for maintaining constraints

Each handle attached to a skeleton knows how to compute a function of the skeleton's state, $\mathbf{h}(\mathbf{q}_{skeleton})$. A number of handle types can be defined, each computing a different quantity. These might include

- A *point handle* which computes the global position of some point defined within a local joint coordinate system. (e.g. a point on the left shin, 6 inches below the knee)
- An *orientation handle* which computes the global orientation of a local joint coordinate system.
- A *center-of-mass handle* which computes the global position of a skeleton's center of mass, computed as a weighted average of each body part's center of mass.

Usually a single handle will depend on only a subset of the joints within a skeleton. When queried for its state vector, a skeleton should return only those elements of $\mathbf{q}_{skeleton}$ which are referenced by a handle. This will keep the total number of variables in the constrained system to a minimum.

In addition to the function $\mathbf{h}(\mathbf{q}_{skeleton})$, each handle must also know how to compute the Jacobian matrix $\mathbf{J_h}(\mathbf{q}_{skeleton}) = \frac{\partial \mathbf{h}}{\partial \mathbf{q}_{skeleton}}$. The constraint solver will query each handle for this information when assembling the global constraint Jacobian matrix $\mathbf{J_C}$. Implementing a new handle type is straightforward, requiring only 3 handle-specific routines to be coded. These routines are responsible for initializing the handle, computing the handle function $\mathbf{h}$, and computing the handle's Jacobian matrix $\mathbf{J_h}$.

5.3.3 Constraints on Handles

Constraints can be applied to the quantities computed by a handle. The simplest constraints are those that constrain a handle to a geometric constant. Useful examples of this include

- constraining a point handle to a given location
- constraining a point handle to a plane
- constraining a point handle to a line
- constraining an orientation handle to a given orientation

These simple constraints would be useful for specifying interactions with a static environment. For example, the feet of a figure can be constrained to lie in the plane defined by the floor, or the hand can be constrained to the rung of a ladder. To specify more complex behaviours, constraints could also be created between two handles. For example, a figure's hands could be constrained to stay clasped together by creating a point handle on each hand, then specifying an equality constraint between these two handles.

Each constraint computes a function $\mathbf{c}(\mathbf{h}_1, \cdots, \mathbf{h}_n)$, where the input values are the outputs of one or more handles. These constraint functions are usually quite simple, often reducing to just a variation of the distance formula. In addition, each constraint must be able to compute its Jacobian matrix with respect to its inputs, $\frac{\partial \mathbf{c}}{\partial \mathbf{h}}$. Adding a new constraint is just a matter of writing code to evaluate these two items. Geometric constraints are for the most part simple enough that this is a minor amount of work.

5.3.4 The Global Picture

An application sets up constraint problems by creating handles on skeletons, imposing constraints on these handles, and applying forces to them before invoking a solver to evaluate equation (5.7). But adding the constraint enforcement scheme of Section 5.2 to an interactive program is not quite straightforward. A user should be able to interactively impose temporary constraints on a figure while editing its posture. Over the course of a normal work session, numerous constraints may be created and deleted as the work proceeds. So the system being simulated by equation (5.7) is constantly changing; each time a constraint is added, new handles may be referenced and new state variables introduced. It is not possible to determine beforehand the correct set of equations to be simulated, and to simply "hard-wire" the code to do so. Instead some sort of mechanism must exist for assembling and evaluating the system of equations on the fly, based on the current set of active constraints. The constraint solver itself can be made responsible for this. It will maintain a *data-flow network* which represents the system being simulated; the equations of motion for the system are completely determined by the network's configuration. The data-flow network scheme has been described by Witkin [AW90], and explored more fully by Kass [Kas92].

The solver is responsible for assembling and solving the global constrained system of equations, taking an integration step, and communicating new state information back to each skeleton object. It also handles any bookkeeping required to map local skeleton variable indices to global indices. As an application adds constraints to the system, the data-flow network within the solver is updated. The network consists of a set of *function blocks*, with outputs of some blocks feeding into the inputs of others. Each block computes two items: a function of the block's inputs, $\mathbf{f}(\mathbf{x})$, and the local Jacobian matrix of that function with respect to its inputs, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$. The block outputs $\mathbf{f}(\mathbf{x})$, as well as the result of multiplying the local Jacobian matrix by an incoming Jacobian matrix (thus applying the chain rule). Figure 5.3 depicts a typical function block.

Each constraint and each handle within the system contribute a single function block to the network. At the top level, the inputs of a constraint block are "wired" to the outputs of one or more handle blocks (or, in the case of simple constraints, directly to constant values). At the bottom level, the inputs to each handle block are connected directly to elements in the global state vector $\mathbf{q}$. As

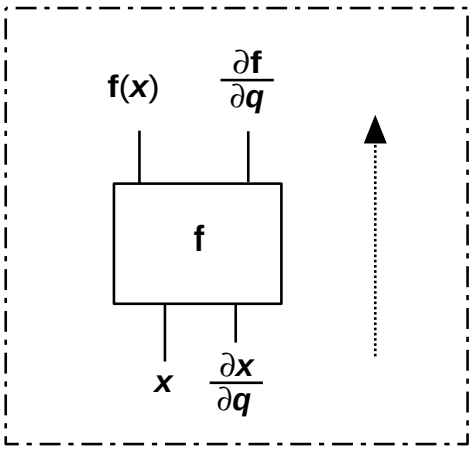


Figure 5.3: A generic function block.

constraints are created and destroyed, nodes are added to and deleted from the network, which is "rewired" to reflect the change.

When the solver is invoked, it consults the top level of the network to determine the global constraint vector $\mathbf{C}$ as well as the constraint Jacobian matrix $\mathbf{J_C}$. The constraint vector $\mathbf{C}$ is formed by simply instructing each top level node to evaluate its function, and concatenating the results. Each node recursively instructs its inputs to evaluate themselves before computing its own function. The recursion bottoms out at the lowest level when it reaches the state variable values $\mathbf{q}$. Each node caches its last computed value in case it is needed again.

A similar recursive scheme is used to compute the global constraint Jacobian matrix $\mathbf{J_C}$. Each top level constraint node computes its own local Jacobian matrix with respect to its inputs, then applies the chain rule by multiplying this against the Jacobian matrix of each of its inputs with respect to the state variables (see Figure 5.3). The solver's recursion scheme takes care of the mapping between local and global indices, as well as the allocation of temporary memory for intermediate matrices. Matrix sparsity is preserved and exploited, to reduce computation, and caching is again used to avoid duplicate computations.

The data-flow network provides the flexibility required to support the interactive restructuring of a simulated system. One attractive feature of this organization is that implementing new classes of handles and constraints is particularly simple. All that is required is to implement the two routines required by a function block, since this is all the solver needs as it traverses the network.

An Example

Consider the problem of constraining the tips of a figure's hands to its hips. This can be accomplished by first creating four point handles on the figure:

- handle $\mathbf{h}_1$ on the tip of the right hand
- handle $\mathbf{h}_2$ on the right hip
- handle $\mathbf{h}_3$ on the tip of the left hand
- handle $\mathbf{h}_4$ on the left hip

To fix the hands to the hips, two equality constraints on these handles are created

$$\begin{aligned} \mathbf{c}_1 &= (\mathbf{h}_1 - \mathbf{h}_2)^2 \\ \mathbf{c}_2 &= (\mathbf{h}_3 - \mathbf{h}_4)^2 \end{aligned}$$

Figure 5.4 shows the resulting data-flow network constructed by the constraint solver. Each handle references a few variables within the skeleton's state vector; these variables are concatenated to form the global state vector $\mathbf{q}$. The handle function blocks compute their respective point locations and feed the results to the constraint function blocks. These compute the distances between the incoming points, and the block function outputs are concatenated to form the 6-element [3] global constraint vector $\mathbf{C}$. Each block also computes its own local Jacobian matrix with respect to its inputs, multiplies it with an incoming matrix and passes the result along. The matrix outputs of the constraint blocks contribute sparse rectangular blocks to the global constraint Jacobian $\mathbf{J_C}$.

Any time an element of $\mathbf{q}$ changes value, a "disable cache" signal is triggered on any handle block connected to the element. This signal percolates up through the network, forcing any block that is encountered to discard its cached data. The next time the solver evaluates the network, only these blocks will recompute their outputs. When the network has been re-evaluated, the solver has all the information it needs to evaluate equation 5.7.

5.3.5 Summary

This constraint enforcement procedure has some attractive characteristics. Most importantly, constraints can theoretically be maintained exactly, since any forces acting on the system which would cause a constraint violation are removed. The procedure is quite general, as well; constraints and handles can be arbitrarily complex functions, and are therefore not restricted to referring to geometric quantities alone. The data-flow network architecture is extensible, since adding new handle and constraint types is simply a matter of writing a handful of functions.

[3] 6 for this example: two sets of x, y, and z scalar constraints.

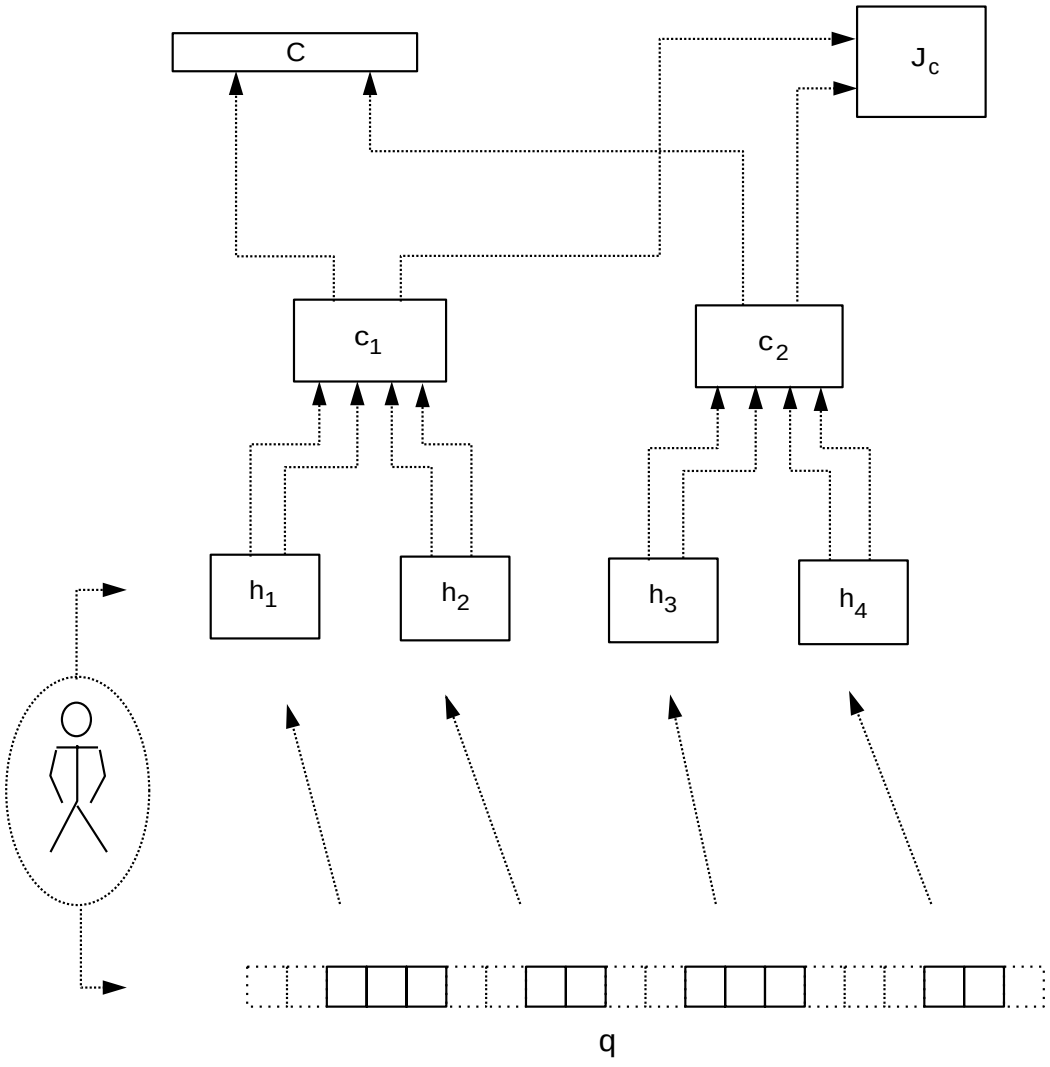


Figure 5.4: Example network.

The main drawback to the approach is that the solution cost is dominated by the $O(n^3)$ cost of solving a linear system for the Lagrange multipliers, of which there is one for each of n scalar constraints in the system being simulated. For even a few simple geometric constraints, the solution time at this bottleneck quickly degrades to the point where interaction becomes difficult. But Surles [Sur92] has shown that when constraints are applied to articulated structures with specific characteristics, the resulting system of equations can be solved in linear time, under certain conditions. This result is promising, but is recent enough that it has not been pursued for this thesis.

A sample implementation of the method outlined above both confirms its potential as well as the limitations imposed by the bottleneck. Retaining interactive update rates for any reasonably complex skeleton, with just a few position and orientation constraints applied, is difficult given current CPU performance. Simple Euler integration, not surprisingly, introduces numerical instabilities, due to cumulative errors, when the integration stepsize is large enough to provide reasonable interactive feedback. Adopting more robust adaptive integration methods is not necessarily helpful either. These methods typically require multiple evaluations of equation (5.7) per iteration, and since it is so expensive to compute, the refresh rate drops dramatically and the interface tends to feel unresponsive. In addition, adaptive stepsizing implies that some steps will be thrown away entirely when the stepsize must be reduced in order to retain accuracy. As a result, one iteration may take much longer to complete than another, so screen updates occur at irregular intervals and consequently the interface is perceived as being "jerky" and unresponsive. However, these are problems which will disappear as CPU performance improves to the point where accuracy in the solution can be maintained at interactive update rates.

Another source of potential stability problems is the feedback term of equation (5.7). This term suffers from the usual problems associated with spring-based constraint enforcement: a high spring constant k leads to a stiff set of equations which are unstable, while a low value for k permits noticeable constraint violations. Choosing a good spring constant value can be difficult, and is currently done purely on a trial-and-error basis. Ideally the feedback spring should be a loose one (i.e. k is "small"), with the assumption that as CPU performance improves it will be possible to perform enough iterations between screen updates to eliminate any noticeable constraint violations. Trying to eliminate the inevitable "drifting" of a system away from its constraints, a result of numerical inaccuracies, by using a high spring constant is almost sure to introduce severe stability problems.

Given current CPU speeds, this technique seems better suited to solving constraint problems off-line than in an interactive setting. Our goal is to develop immediately useful interactive tools for working with non-trivial skeletons, and this method cannot be considered practical yet for this

purpose on modestly powered workstations. Instead we will consider a penalty-force method based on the CCD algorithm as a substitute with which we can experiment in an interactive editor.

5.4 A CCD-based Penalty Method

The previous method is capable of handling multiple constraints imposed on different parts of a skeleton. To accomplish the same with the CCD method requires some slight modifications to the basic algorithm presented in Chapter 4. Whereas the basic algorithm considered just a serial chain of links and a single goal, solving for multiple constraints means we must also consider branches in the manipulator and more than one goal, as shown in Figure 5.5.

The scheme of Chapter 4 solved the problem for a serial chain by traversing from the end-effector in towards the base, at each joint adjusting the joint parameter to minimize an error measure derived from the current end-effector position and a known goal. In a branching chain with multiple end-effectors and multiple goals, a joint may contribute to the position of more than one end-effector, so potentially more than one goal must be considered when varying the joint variable. Also, the algorithm depends on the inward traversal scheme; distal joints must be solved first before a more proximal one can be considered. This precludes treating the multiple-goal problem as a series of single-goal problems which can be solved sequentially.

A recursive traversal of the hierarchy from the leaves in towards the root ensures that all sub-problems at a joint are solved before the joint itself is considered. Each joint instructs all of its children to solve for any constraints that may be applied to end-effectors in their sub-tree. Then the joint variable which minimizes the summed error measure from each child sub-tree can be computed. The only change from the previous CCD algorithm is that the traversal must now consider branches

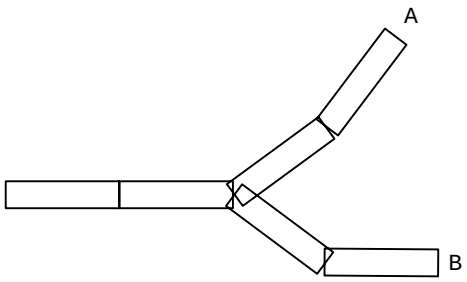


Figure 5.5: A branching chain with two end-effector constraints

in a chain, and that the error measure equation (4.13) of Chapter 4 becomes a summation

$$E(\mathbf{q}) = \sum_{i=1}^{n} E_{p_i}(\mathbf{q}) + E_{oi}(\mathbf{q}) \tag{5.8}$$

over the n constraints which are distal to the current joint. The net effect of this change is that the coefficients k_1, k_2, and k_3 must be computed by summing equations (4.20)-(4.22) over each distal end-effector goal, before a minimizing change in the current joint variable can be computed. The procedure must iterate until either all end-effector constraints have been satisfied, or until successive iterations produce negligible changes, indicating that the constraints are conflicting and cannot be simultaneously satisfied.

This recursive scheme resembles Badler's early heuristic approach [BMW87] for positioning a figure with multiple constraints, although the CCD method for computing a minimizing change in a joint variable appears to differ from the one used there.

This penalty method approach is arguably inferior to the more comprehensive scheme outlined previously. Constraints are only approximately enforced, so we can expect to see some "drift" on segments that are constrained to stay in place. Also, we are restricted to geometric constraints due to the CCD method's reliance on geometric relationships alone. Nevertheless the method is usually stable enough to perform adequately at interactive update rates, and gives us a means of experimenting with constraints within our interactive editor. In our prototype implementation, instabilities do occur when position and orientation constraints conflict with each other, but this can be alleviated with appropriate weighting factors on the individual constraints.

Chapter 6

An Interactive Editor

To place the discussion of the preceding chapters in context, an interactive program for animating articulated figures is introduced here. The program is designed to create keyframed motion *sequences* for arbitrary skeletons, and has been both a motivation and a testbed for the ideas developed in this thesis. Both the Jacobian transpose and the CCD methods have been implemented, and the application interface modified to accomodate direct inverse kinematic manipulation, as well as constrained manipulation. This chapter briefly describes the program and the manipulation interface.

6.1 A System Overview

The *Sequence Editor* is an interactive tool for creating and editing keyframed movement sequences for a single arbitrary skeletal figure. Its primary function has been to act as the "movement creation" window of a motion planning package [CWGL89] for choreographers, whose needs are quite different from those of computer animators. To a typical user planning the movement of multiple figures in a work space, a rough approximation of a movement that can be created quickly can be more valuable than a more realistic animation that would take hours to perfect. Consequently the program's design has favoured ease of use over functionality. The hope is that direct manipulation and constraint satisfaction based on the methods of the previous chapters can improve the quality of movement created with the editor while retaining a simple and intuitive user interface.

Before describing the interface itself, some of the terminology warrants a brief explanation.

6.1.1 Skeletons

The program itself knows nothing about skeletons specifically; a skeleton appears as an abstract data type supported by a toolkit library of routines. The toolkit supports the creation, manipulation and

animation of skeletons defined by a simple ASCII file format; Figure 6.1 [1] shows a sample description for a simple approximation to a human figure. Any valid skeleton description can be read in to the editor and animated.

As the example shows, each skeleton is simply a collection of named joints arranged in a hierarchy. The joints are either rotary or prismatic, each with a single degree of freedom. Compound joints with multiple degrees of freedom (e.g. ball-and-socket) can be modelled as a series of these single DOF joints sharing the same location. Associated with each joint are a number of attributes, including the joint type (rotary or prismatic), the local axis (axes) of freedom, and local limits on the range of movement of the joint. Polygonal objects may be attached to any joint node in the hierarchy. These (rigid) objects comprise the appearance of the skeleton and are drawn as they are encountered during a display traversal of the hierarchy.

A skeleton maintains an internal state vector $\mathbf{q}$, consisting of all the joint variable values as well as a translation vector for the skeleton as a whole. This information is enough to completely define the location and shape of the skeleton for a single frame. An application can animate a skeleton by repeatedly loading the skeleton state $\mathbf{q}$ and instructing the skeleton to display itself.

The toolkit now supports inverse kinematic control over a skeleton by allowing the application to identify both a base and an end-effector within the joint hierarchy. A desired position and/or orientation for the end-effector may then be specified, and the skeleton instructed to solve the inverse kinematic problem using either of the methods of the previous chapter. The internal state $\mathbf{q}$ is automatically updated to reflect the solution to the problem. This isolates the application from the details of the inverse kinematic algorithms.

6.1.2 The Sequence

A *sequence* is an abstract data type for storing skeleton animation data; it represents the movements of a skeleton over some period of time. An application may query a sequence to determine the skeleton state at any frame of the animation. Although a sequence could encapsulate a procedural motion model, for this discussion a sequence will be defined as a series of keyframed poses, each of which specifies the state vector $\mathbf{q}$ at a particular frame. When a sequence is queried for the skeleton state at a frame between key frames, an interpolated state is computed on-the-fly and returned. For each rotating joint, either linear or splined quaternion [2] interpolation [Sho85] may be performed. For each translating joint, either linear interpolation or generalized Catmull-Romspline interpolation [KB84] may be used. Currently, no further control over the interpolation process is provided, other

[1] courtesy of Phil Peterson

[2] a quaternion is a compact representation for rotations, with some appealing properties

```
# Example .scr file for a stick figure.

define skeleton "stickman"
    joint "pelvis"
    group "Upper body"
        joint "torso"
        group "Right Arm"
            joint "right arm"
            joint "right hand"
        end group
        group "Left arm"
            joint "left arm"
            joint "left hand"
        end group
        joint "head"    # "head" is a child of "torso"
    end group
    group "Right leg"
        joint "right leg"
        joint "right foot"
    end group
    group "Left leg"
        joint "left leg"
        joint "left foot"
    end group
end skeleton
```

```
define joint "right leg"
    object "r_lleg.pol"
    offset 0 -0.32 0
    hinge x 0 170
    mass 1
    orientation 0 0 0
    mirror "left leg"
end joint
```

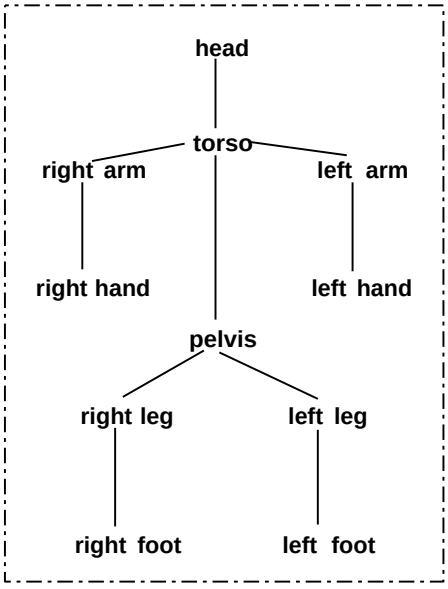


Figure 6.1: Sample skeleton description

than that afforded by changing the keyframe spacing or the key poses.

A number of simple editing operations are defined for keyframe sequences. Ranges of keyframe poses may be copied, inserted, and/or deleted. Keyframe spacing may also be adjusted by stretching, shrinking, or sliding (moving) a range within the sequence. This provides some crude control over timing.

6.1.3 The Editor

An interactive editor for creating and editing animation sequences for arbitrary skeletons is shown in Figure 6.2. The current skeleton is displayed in three orthogonal views and a single perspective view. Each of these views may occupy the large main work area, in which the skeleton may be manipulated. Existing sequences are grouped into menus and displayed in iconic form to the right

Figure 6.2: Sequence editor screen

of the screen. Each sequence icon can cycle through all of the keyframe poses on request, generating a simple flipbook preview of the movement as a memory aid. A sequence can be named and added to one of these menus, and a menu item can later be reloaded for editing. A small display beneath the main viewport shows the keyframe poses within the loaded sequence. These are laid out along

a *timeline* to indicate the keyframe spacing. A set of VCR-like transport controls are provided for playing back the animated sequence in the main display area.

The small timeline provides a convenient interface for copying, cutting and pasting ranges of keyframes within a sequence, as well as modifying the keyframe spacing. But the key poses themselves must be created or modified by adjusting the skeleton displayed in the main window. For manipulating the skeleton, forward kinematic controls are provided in the form of sliders for adjusting individual joints within the skeleton one at a time. In addition, the interface has been modified to permit inverse kinematic manipulation of the skeleton. In this mode the user may use the methods presented in the previous chapter to update multiple joints by dragging one body part around. The implementation of this interface for direct manipulation is briefly described below.

6.2 Direct Manipulation

With direct manipulation, the user adjusts the skeleton by selecting and dragging a body part, rather than adjusting individual joints. A dragging session begins when a body part is selected with the mouse, and ends when the mouse button is released. Dragging is a positioning mechanism only, it does not directly control the orientation of the selected part. To support dragging, the interface must provide a way of identifying a serial *chain* of joints within the skeleton which will be considered a manipulator, as well as a way of unambiguously specifying end-effector goals for the chain.

Identifying the Chain

A chain of joints within the skeleton is defined when a body part is first selected, at which time both a base joint and an end-effector must be determined. To determine a base joint, the skeleton hierarchy is traversed backwards from the selected part towards the root. Three separate interaction modes control how far up the tree this traversal proceeds before stopping at a joint which becomes the base of the chain. The stopping criteria corresponding to each of the three dragging modes are

1. stop when the parent joint has a body part attached to it
2. stop when the parent joint has multiple children
3. stop when a named joint is reached

The first of these results in only the selected body part being dragged, and is the direct manipulation equivalent of positioning the part using the forward kinematic slider controls. The second results in a chain which extends from the nearest branch in the hierarchy, and is therefore a useful default to use when dragging limbs. Selecting and dragging a hand, for example, would move the entire arm without disturbing the torso. The third interaction mode permits the user to override these default

behaviours by explicitly naming a joint from which the chain extends. Thus the user can specify a torso joint as the chain base, for example, in which case pulling on the arm would result in a bend of the spine.

In addition to the base of the chain, the end-effector position must be known, since that is the point being dragged. There are two possible approaches to determing the end-effector location when a user points at a body segment and presses the mouse button. The first is to explicitly compute the point on the body segment underneath the cursor, by casting a ray and intersecting it with the object. The alternative is to pre-define locations within the skeleton which may be selected for dragging; the locations of the joints themselves, for example. This requires either that the locations be displayed in some iconic form so the user can select one, or that the cursor "warp" to the nearest one when the mouse button is pressed. The initial decision was to choose the former method: explicitly computing a point on the surface of the object and making that point the location of the end-effector frame. This gave mixed results. On one hand it allowed the user to click anywhere on the object for dragging without any disconcerting cursor "warping". On the other hand, having the end-effector frame located on the object surface often resulted in unexpected twisting of the segment, especially with the Jacobian method. As a compromise, the current implementation places the end-effector halfway along the segments axis of self-rotation, "warping" the cursor to the corresponding onscreen location. Locating the end-effector along the segment axis eliminates the problem of unexpected twisting.

Determining an End-Effector Goal

To compute new goal positions for the end-effector as the cursor moves around on the screen, we must resolve the ambiguity in mapping a 2D screen location to a 3D location. The solution adopted here is to cast a ray through the cursor into the world, and to find the point on this line which is closest to the end-effector. This point becomes the goal position for the current iteration. This technique provides reasonable behaviour in both orthographic and perspective views.

With an orthographic projection, the cast ray is parallel to the line of sight, and the closest point on the line will lie in the plane perpendicular to the ray and containing the end-effector. In this case the end-effector is constrained to lie within the plane while it is dragged around. In a perspective projection view, the cast ray may diverge from the line of sight, and the shortest path from the end-effector to the ray is no longer constrained to lie parallel to the image plane. The end-effector is therefore free to move in any direction, and may even be "pushed" and "pulled" towards and away from the viewer with a little practice. Figure 6.3 illustrates the case for both perspective and orthographic projections.

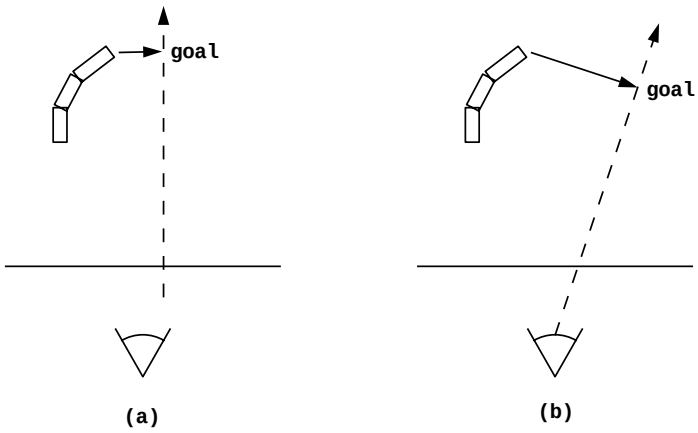


Figure 6.3: Plan view of goal determination in (a) an orthographic view, and (b) a perspective view

Once the new goal position for the end-effector has been computed, the skeleton is instructed to solve the inverse kinematic problem; the user can select the solution method to use by setting a program option. The screen is refreshed and the cursor location resampled after each iteration, to provide responsive feedback. Of course, when a single iteration is insufficient to solve the most recent problem, the movement of the skeleton lags behind the cursor. In practice this small tracking delay has not been a problem.

To provide some control over the responsive behaviour of the skeleton, the user can interactively modify weighting parameters at each individual joint. This is particularly useful when the CCD method is used, allowing a the responsiveness of a chain to be varied from that of a stiff rod to that of a set of loosely coupled links.

6.3 Constraints

As an additional positioning aid, a small set of constraints have been defined which the user may apply to any segment of a skeleton. The most useful of these are intended to lock limb extremities in place while the pose is being edited. Currently constraints are enforced by iteratively solving a set of persistent inverse kinematic goals using the CCD method. This restricts us to a set of simple geometric goals:

- Locking an end-effector position
- Locking an end-effector orientation. This can be a partial lock of only one or two directions (e.g. lock Y orientation only).

- A weighted combination of both of these
- Constraining an end-effector to lie within a plane

To apply a lock the user selects a segment, then makes a lock selection from a menu. Once created, a lock remains in place until it is explicitly deleted. A lock can be edited, to change either its type, its importance weighting, or the joints which can contribute to maintaining the lock. The lock location or orientation can also be modified by either dragging the lock, or twisting the end-effector through a set of sliders.

The intent is that locks represent constraints which are to be maintained during any subsequent positioning of the figure. To achieve this, after each editing operation of a figure, whether through direct manipulation or the forward kinematic slider controls, a number of iterations of the multiple-goal CCD solver are performed prior to each screen refresh. When the number of iterations performed is insufficient to satisfy any constraints violated by the editing operation, some drifting from the lock constraint positions is unavoidable. But a button is provided for invoking the solver explicitly until all constraints are met to the user's satisfaction. The user may also change the number of iterations performed between screen refreshes. On a high-performance machine this number can be set high to minimize visible constraint violations; on a low-performance machine a smaller number can improve the responsiveness of the interface, at the cost of constraint violations becoming more noticeable.

Locking constraints are particularly useful when applied to the supporting limbs of a figure. In the example of Figure 6.4 the figure's feet are constrained to stay in place by locking both their position and orientation. The first image shows the position at which the locks are created. In the second image the pelvis has been tilted to lean the figure, and in the third image a twist has been applied to the pelvis. In all three images the foot positions and orientations are maintained, and the CCD solver is fast enough that the pelvis can be tilted and twisted interactively without noticeable sliding of the feet. For this example, 5 iterations between screen refreshes were sufficient to maintain the constraints with negligible drift, and provide an update rate of 3-4 frames per second on a Silicon Graphics R3000 Entry Level Indigo workstation. This update rate is inadequate for animation, but quite sufficient for interactive positioning.

While locking end-effectors in position is useful for creating a series of consistent keyframe poses, there still remains the problem of maintaining the constraints at the interpolated in-between frames. There is no guarantee that an end-effector locked to some position in two adjacent keyframes will remain in that position if joint angles are merely interpolated between the poses. Figure 6.5 illustrates the problem. The first and last images are keyframes in which the feet have been constrained to the same position and orientation. The image in the middle is the result of interpolating joint

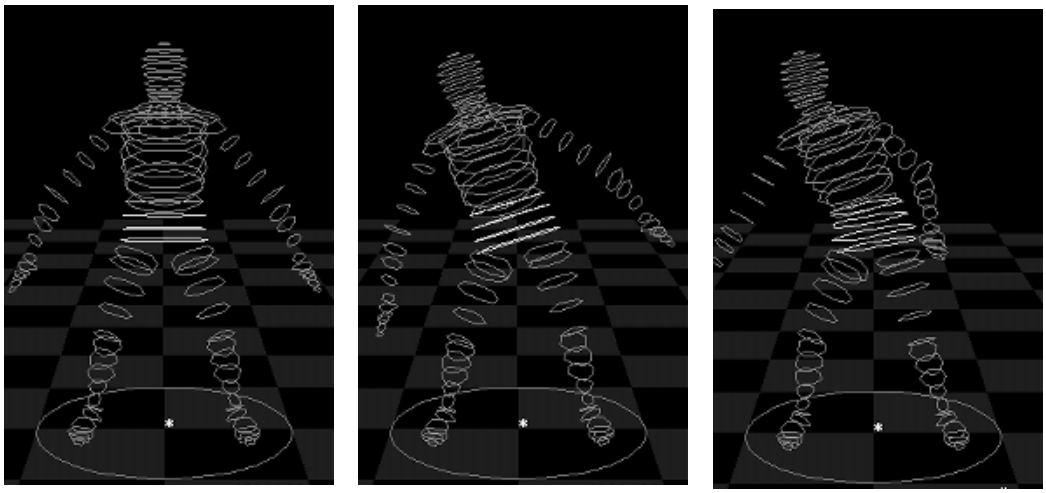


Figure 6.4: A figure being positioned by first tilting, then twisting, the pelvis.

angles between the two poses; the feet clearly move through the floor during the transition between the keyframes. The problem is that the figure's hierarchy is rooted at the pelvis, and there is no relationship between the rate at the which the pelvic translation is being interpolated and the rate at which the leg joint angles need to change to maintain the foot positions. Of course, we could rearrange the hierarchy to root it at one of the feet, but this would only alleviate the problem for that foot; rearranging the hierarchy is not helpful when multiple end-effectors are constrained.

In the current implementation the only attempt made to address this problem is to provide an option to enable constraint satisfaction during interpolation. Each frame is computed as usual by interpolating joint angles, then the CCD solver is invoked until any violated constraints have been resatisfied. In essence a series of single-frame constraint problems are solved. This approach is admittedly *ad hoc*, and it is not clear yet how well it will work; it appears adequate for the few situations tested so far, although interpolation can no longer be performed on-the-fly.

Initial feedback suggests that even this simple approach to enforcing geometric constraints, combined with direct inverse kinematic manipulation, is a significant improvement to the keyframe editor.

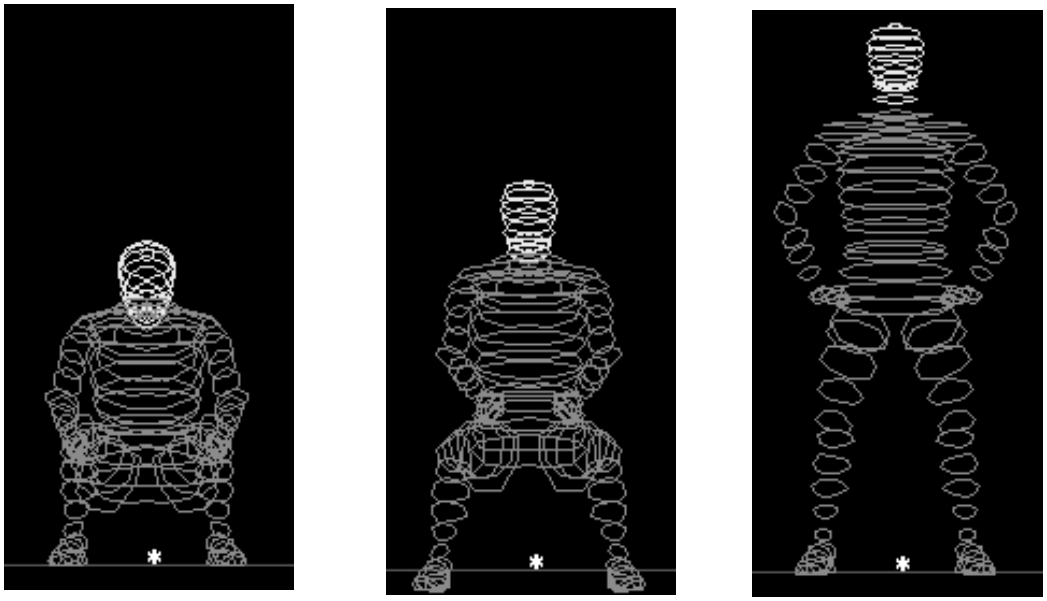


Figure 6.5: (a) First keyframe pose (b) Interpolated pose (c) Second keyframe pose

Chapter 7

Conclusion

7.1 Summary

We have examined solutions to the inverse kinematic problem applied to manipulating articulated skeletons in an interactive keyframe animation editor. As alternatives to previously published algorithms in the graphics literature, a pair of simple algorithms have been described which can provide relatively inexpensive direct manipulation of a figure. The first of these is really just an application of a simple minimization method, but its application to inverse kinematics has not previously been made explicit. The second is a new heuristic algorithm adopted and modified from the robotics literature. The advantages and disadvantages of each have been discussed and their relative performances gauged.

Methods by which each algorithm can be used to satisfy constraints during manipulation have been developed, and their respective advantages and disadvantages discussed. An implementation of the simpler approach has been incorporated into a keyframe editor, and the interface briefly described.

As a byproduct, a toolkit library for defining and manipulating skeletons has been implemented which insulates an application from the details of creating, editing, drawing and animating a figure. A skeleton is also able to solve inverse kinematic problems posed by the application, allowing the application to deal with interface issues rather than dealing explicitly with inverse kinematic algorithms. This also allows new inverse kinematic solution methods to be implemented and added to the toolkit without requiring modifications to the application.

7.2 Results

After experimenting with several inverse kinematic methods in the course of this work, some things have become apparant. The first is that inverse kinematic algorithms all exhibit problems of one type or another; no one approach seems uniformly superior to others. The methods presented here should not necessarily be considered any better than other approaches, but rather provide alternative approaches which may be suitable in some applications.

The second point is that inverse kinematics as a positioning tool is a useful complement to, but not a replacement for, simple forward kinematic positioning. In particular, for direct manipulation inverse kinematics is really only useful for chains with relatively few degrees of freedom - single limbs, for example. In fact, an interesting observation is that even with the ability to drag multiple-jointed chains around many users of the editor described in Chapter 6 seem quite content with the default dragging mode, which drags just a single body part. Informal feedback from users suggests that while direct manipulation is a huge improvement to the interface over the previous slider-based positioning, the value of multiple-joint positioning is not quite so clear. In fact, trying to position a chain with many degrees of freedom with direct manipulation is often likely to cause unwanted changes to the skeleton. For example, trying to bend the spine of a figure by pulling on its fingers is probably asking too much - in the absence of additional constraints specifying how changes should be distributed among the joints, any method is likely to give unnatural results. Joint limits provide some constraints, as do the weighting parameters at each joint, but these do not seem to be adequate, particularly for manipulating recognizable bodies. Inverse kinematic methods which perform quite acceptably for disembodied chains or mechanical robots do not necessarily translate well to "animate" figures; the results can often appear unnatural. This problem is that the term "unnatural" is subjective, and difficult to quantify. More powerful positioning tools are likely to result from considering these factors more carefully.

Either of the inverse kinematic algorithms presented in Chapter 4 is adequate for interactive direct manipulation. They are both simple to implement, and are fast enough to provide good feedback for gross positioning tasks. However if high accuracy is a requirement and interactive response time not so much of a concern, then other solution methods may offer better performance.

7.2.1 Comments about Constraints

In Chapter 5, the choice of the CCD-based penalty method for implementation within the figure animation editor was largely based on performance issues; the alternative Lagrange multiplier-based approach was deemed too demanding for interactive use on a typical workstation. This problem is temporary, however, and it is worth reconsidering the decision in light of continuing increases in

CPU power. At the same time we can speculate a little about what types of constraints might be useful to develop and how constraints might be used for animation as well as manipulation.

Given sufficient CPU speed to retain numerical stability, the Lagrange multiplier approach is the more general of the two. The CCD-based method is fast because it reduces a difficult problem to a series of simpler ones which can be solved analytically using only simple geometric quantities. But for problems more complicated than solving position and orientation constraints the reduction to analytical form is not always possible, so the approach is limited to these types of constraints. In contrast the Lagrange multiplier method could be used to enforce many different types of constraints, including non-geometric ones; the only requirement is that the constraints be some function of the joint variables within a figure.

Looking Ahead

What sorts of constraints, other than the obvious geometric ones, might be useful for figure manipulation? Initial experimentation with the constraint solver implemented in the animation editor indicates that geometric constraints alone, while useful, are not always enough. Too often a solution which is correct in the sense that all geometric constraints are satisfied, seems incorrect because it puts the figure into an awkward looking pose. Think, for example, of a person going from a standing position into a crouch. The natural tendency is for the knees to move apart as the person crouches, because that is the comfortable way to crouch. To manipulate a standing figure in the editor into a crouch, one might lock the toes of each foot in place and then pull the pelvis down to make the figure crouch. But without any other information about how the legs should move, one is just as likely to see the knees move together as the figure crouches, as to see them move together. Perhaps this could be rectified by specifying additional geometric constraints on the knees, but choosing and specifying these auxiliary constraints would probably be difficult and time-consuming. A useful alternative would be a standard set of constraints which applied to the figure at all times. These might include constraints to avoid uncomfortable postures, or postures which would place a person off-balance. The latter can be implemented as a constraint on a figure's center-of-mass, a computable quantity given a set of joint angles, while the former would require some function which measured comfort, or "naturalness", given the same set of variables.

In addition to manipulation, constraints could also be useful in animating a figure. A natural place to start would be to consider animating the constraint values. Animating a position constraint, for example, might define a trajectory for some limb extremity to follow. A simultaneous animated orientation constraint would control the orientation of the extremity as it traversed the trajectory. But for complex motions, constraints will probably be more helpful in describing a desired movement with some sort of scripting approach. For example, the movements of the support foot of a walking

biped might be described in terms similar to the following

> "After heel strike, the heel of the foot maintains its position until the ball of the foot touches the ground. For a short period the entire foot remains locked in place while the body moves forward. Toward the end of the step, the heel of the foot breaks contact with the ground first, followed shortly after by the ball of the foot."

A description such as this identifies a number of simple constraints which are in effect during the movement. If the description were more specific about the time intervals and locations involved we would have a fairly detailed script for animating the stance foot during a walk. A simple scripting language could be implemented which would allow movements such as this to be described textually. A script would need to specify a set of constraints to be imposed, and provide some way of activating and deactivating these constraints over the course of a movement. An interpreter for such a descriptive scripting language would be a useful tool for figure animation.

7.3 Directions

There are plenty of problems still to be addressed. At a low level of detail it is worth determining whether the performance of the Lagrange multiplier constraint satisfier can be improved upon. First, Surles has shown that some constraint problems for articulated figures can be solved in linear time [Sur92]. He lists a number of prerequisites for this, but in the context of chemical protein modelling, which is his application area. An analysis of these prerequisites and what they mean in the context of articulated skeleton manipulation would be useful. A second area which could be explored to speed up the solution process is to consider Maciejewski's incremental algorithm for computing the SVD [MK89]; the referenced papers on this topic are interesting reading, and may suggest alternate approaches to the inverse kinematic problem in general.

Designing interactive interfaces which make effective use of constraints is challenging, and deserves some attention. How should constraints be represented? Selected? Edited? What should happen when constraints conflict with each other? How should constraints be animated and controlled over time? What about constraints involving multiple figures?

And finally, do inverse kinematics and constraints provide a convenient layer upon which higher level procedural motion models can be built? An interesting exercise would be to extend Bruderlin's walking algorithm to handling uneven terrain, making use of the basic techniques described in this thesis to achieve and maintain footholds.

Bibliography

[AG85] W. Armstrong and M. Green. The dynamics of articulated rigid bodies for purposes of animation. *The Visual Computer*, 1:231–240, 1985.

[AN88] L. Alt and A. Nicolas. Animating articulated structures with multiple goals. In *Proceedings of Computer Graphics*, pages 215–225, 1988.

[AW90] W. Welch A. Witkin, M. Gleicher. Interactive dynamics. *Computer Graphics*, 24(2), March 1990.

[Bai86] J. Bailleiul. Avoiding obstacles and resolving kinematic redundancy. In *Proc. 1986 Int. Conf. on Robotics and Automation*, April 1986.

[BB87] R. Barzel and A. Barr. Modeling with dynamic constraints. In *SIGGRAPH Course Notes: Topics in Physically-Based Modelling*, 1987.

[BC89] A. Bruderlin and T. Calvert. Goal-directed dynamic animation of human walking. *Computer Graphics*, 23(3):233–242, July 1989.

[BH89] R. Bartels and I. Hardtke. Speed adjustment for keyframe interpolation. In *Proceedings of Graphics Interface*, pages 14–19, 1989.

[BMW87] N. Badler, K. Manoochehri, and G. Walters. Articulated figure positioning by multiple constraints. *IEEE Computer Graphics and Applications*, 7(6):28–38, June 1987.

[BN88] L. Brotman and A. Netravali. Motion interpolation by optimal control. *Computer Graphics*, 22(4):309–315, August 1988.

[CHP89] J. Chadwick, D. Haumann, and R. Parent. Layered construction for deformable animated characters. *Computer Graphics*, 23(3):243–252, July 1989.

[Coh92] M.F. Cohen. Interactive spacetime control for animation. *Computer Graphics*, 26(2), July 1992.

[CP90] N. Badler C. Phillips, J. Zhao. Interactive real-time articulated figure manipulation using multiple kinematic constraints. In *Proceedings, 1990 Symposium on Interactive 3D Graphics*, pages 245–250, 1990.

[CWGL89] T.W. Calvert, C. Welman, S. Gaudet, and C. Lee. Composition of multiple figure sequences for dance and animation. In *Proceedings of CG International*, 1989.

[DSS88] H. Das, J.J. Slotine, and T.B. Sheridan. Inverse kinematic algorithms for redundant systems. In *Proceedings IEEE Int'l. Conference on Robotics and Automation*, pages 43–48, 1988.

[FW88] D. Forsey and J. Wilhelms. Techniques for interactive manipulation of articulated bodies using dynamic analysis. In *Proceedings of Graphics Interface*, pages 8–15, 1988.

[Gir86] M. Girard. Interactive design of 3d computer-animated legged animal motion. In *Proceedings, Workshop on Interactive 3D Graphics*, pages 131–150, 1986.

[Gir91] M. Girard. Constrained optimization of articulated animal movement in computer animation. In N. Badler, B. Barsky, and D. Zeltzer, editors, *Making Them Move: Mechanics, Control, and Animation of Articulated Figures*, chapter 10, pages 209–229. Morgan Kaufmann Publishers,Inc., San Mateo, Ca., 1991.

[GM85] M. Girard and A. Maciejewski. Computational modeling for the computer animation of legged figures. *Computers Graphics*, 19(3):263–270, July 1985.

[GMW81] P. Gill, W. Murray, and M. Wright. *Practical Optimization*. Academic Press, New York, NY, 1981.

[Gom85] J. Gomez. Twixt: A 3d animation system. *Computers and Graphics*, 9(3):291–298, March 1985.

[GP90] B. Guenter and R. Parent. Computing the arc length of parametric curves. *IEEE Computer Graphics and Applications*, 10(3):72–78, May 1990.

[Gre59] T.N.E. Greville. The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations. *SIAM Review*, 1(1):38–43, January 1959.

[GW91] M. Gleicher and A. Witkin. Differential manipulation. In *Proceedings of Graphics Interface '91*, 1991.

[Hah88] J.K. Hahn. Realistic animation of rigid bodies. *Computer Graphics*, 22(4):299–308, August 1988.

[HS85] P. Hanrahan and D. Sturman. Interactive animation of parametric models. *The Visual Computer*, 1:260–266, 1985.

[IC87] P. Isaacs and M. Cohen. Controlling dynamic simulation with kinematic constraints, behaviour functions and inverse dynamics. *Computer Graphics*, 21(4):215–224, July 1987.

[IC88] P. Isaacs and M. Cohen. Mixed methods for complex kinematic constraints in dynamic figure animation. *The Visual Computer*, 4:296–305, 1988.

[Kas92] M. Kass. Condor: Constraint-based dataflow. *Computer Graphics*, 26(2), July 1992.

[KB82] J. Korein and N. Badler. Techniques for goal-directed motion. *IEEE Computer Graphics and Applications*, 2(9):71–81, 1982.

[KB84] D. Kochanek and R. Bartels. Interpolating splines with local tension, continuity, and bias control. *Computer Graphics*, 18(3):33–41, July 1984.

[KH83] C.A. Klein and C.H. Huang. Review of pseudoinverse control for use with kinematically redundant manipulators. *IEEE Transactions on Systems, Man, and Cybernetics*, 13(3):245–250, March/April 1983.

[Las87] J. Lasseter. Principals of traditional animation applied to 3d computer animation. *Computer Graphics*, 21(4):35–44, July 1987.

[LWZB90] P. Lee, S. Wei, J. Zhao, and N. Badler. Strength guided motion. *Computer Graphics*, 24(4):253–262, 1990.

[Mac90] A. A. Maciejewski. Dealing with the ill-conditioned equations of motion for articulated figures. *IEEE Computer Graphics and Applications*, 10(3):233–242, May 1990.

[MK89] A.A. Maciejewski and C.A. Klein. The singular value decomposition: Computation and applications to robotics. *International Journal of Robotics Research*, 8(6), December 1989.

[NN90] Z.R. Novakovic and B. Nemec. A solution of the inverse kinematics problem using the sliding mode. *IEEE Transactions on Robotics and Automation*, 6(2):247–252, April 1990.

[Pau81] R.P. Paul. *Robot Manipulators: Mathematics, Programming, and Control.* MIT Press, Cambridge, MA, 1981.

[PB91] C. Phillips and N. Badler. Interactive behaviours for bipedal articulated figures. *Computer Graphics*, 25(4):359–362, July 1991.

[PFTV90] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling. *Numerical Recipes in C - The Art of Scientific Computing.* Cambridge University Press, Cambridge, MA, 1990.

[SB85] S. Steketee and N. Badler. Parametric keyframe interpolation incorporating kinetic adjustment and phrasing control. *Computer Graphics*, 19(3):255–262, July 1985.

[Sho85] K. Shoemake. Animating rotation with quaternion curves. *Computer Graphics*, 19(3):245–254, July 1985.

[SS87] L. Sciavicco and B. Siciliano. A dynamic solution to the inverse kinematic problem of redundant manipulators. In *IEEE Int'l Conference on Robotics and Automation*, pages 1081–1086, March 1987.

[Ste83] G. Stern. Bbop - a program for 3-dimensional animation. In *Nicograph Proceedings*, pages 403–404, 1983.

[Stu87] D. Sturman. Interactive keyframe animation of 3d articulated models. In *SIGGRAPH Course Notes: Computer Animation: 3D Motion Specification and Control*, pages 17–26, 1987.

[Sur92] M.C. Surles. An algorithm with linear complexity for interactive, physically-based modeling of large proteins. *Computer Graphics*, 26(2), July 1992.

[SZ88] K. Sims and D. Zeltzer. A figure editor and gait controller for task level animation. In *SIGGRAPH Course Notes: Synthetic Actors: The Impact of Artificial Intelligence and Robotics on Animation*, 1988.

[WC91] L.C.T. Wang and C.C. Chen. A combined optimization method for solving the inverse kinematics problem of mechanical manipulators. *IEEE Transactions on Robotics and Automation*, 7(4):489–499, August 1991.

[WCH88] B. Wyvill, M. Chmilar, and C. Herr. A simple model of human animation. In *SIGGRAPH Course Notes: Synthetic Actors: The Impact of Artificial Intelligence and Robotics on Animation*, 1988.

[WE84] W.A. Wolovich and H. Elliot. A computational technique for inverse kinematics. In *Proceedings of 23rd Conference on Decision and Control*, pages 1359–1362, December 1984.

[Wil86] J. Wilhelms. Virya - a motion control editor for kinematic and dynamic animation. In *Proceedings of Graphics Interface*, pages 141–146, 1986.

[Wil87] J. Wilhelms. Using dynamic analysis for realistic animation of articulated bodies. *IEEE Computer Graphics and Applications*, 7(6):12–27, June 1987.

[Wil91] J. Wilhelms. Dynamic experiences. In N. Badler, B. Barsky, and D. Zeltzer, editors, *Making Them Move: Mechanics, Control, and Animation of Articulated Figures*, chapter 13, pages 265–280. Morgan Kaufmann Publishers,Inc., San Mateo, Ca., 1991.

[WK88] A. Witkin and M. Kass. Spacetime constraints. *Computer Graphics*, 22(4):159–168, August 1988.

[ZB89] J. Zhao and N. Badler. Real time inverse kinematics with joint limits and spatial constraints. Technical Report Technical Report MS-CIS-89-09, University of Pennsylvania, 1989.

[Zel82a] D. Zeltzer. Motor control techniques for figure animation. *IEEE Computer Graphics and Applications*, 2(9):53–59, 1982.

[Zel82b] D. Zeltzer. Representation of complex animated figures. In *Proceedings of Graphics Interface*, pages 205–211, 1982.