CSCI 520 Computer Animation and Simulation

Quaternions and Rotations

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Rotations

- Very important in computer animation and robotics
- Joint angles, rigid body orientations, camera parameters
- 2D or 3D

Rotations in Three Dimensions

- Orthogonal matrices:

$RR^T = R^TR = I$
$\det(R) = 1$

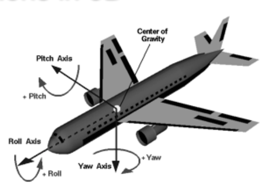
$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Representing Rotations in 3D

- Rotations in 3D have essentially three parameters
- Axis + angle (2 DOFs + 1DOFs)
  - How to represent the axis?
    Longitude / lattitude have singularities
- 3x3 matrix
  - 9 entries (redundant)

Representing Rotations in 3D

- Euler angles
  - roll, pitch, yaw
  - no redundancy (good)
  - gimbal lock singularities



Source: Wikipedia

- Quaternions
  - generally considered the “best” representation
  - redundant (4 values), but only by one DOF (not severe)
  - stable interpolations of rotations possible

Euler Angles

1. Yaw
   rotate around y-axis
2. Pitch
   rotate around (rotated) x-axis
3. Roll
   rotate around (rotated) y-axis

Source: Wikipedia

Gimbal Lock

When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.

Source: Wikipedia

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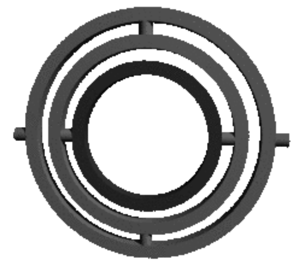

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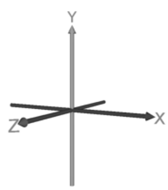
Choice of rotation axis sequence for Euler Angles

- 12 choices:
  XYX, XYZ, XZX, XZY,
  YXY, YXZ, YZX, YZY,
  ZXY, ZXZ, ZYX, ZYZ



- Each choice can use static axes, or rotated axes, so we have a total of 24 Euler Angle versions!

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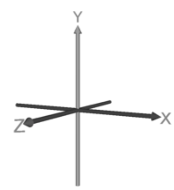
Example: XYZ Euler Angles

- First rotate around X by angle θ_1, then around Y by angle θ_2, then around Z by angle θ_3 .



- Used in CMU Motion Capture Database AMC files
- Rotation matrix is:

$$R = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & \sin(\theta_2) \\ 0 & 1 & 0 \\ -\sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

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Outline

- Rotations
- Quaternions
- Quaternion Interpolation

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Quaternions

- Generalization of complex numbers
- Three imaginary numbers: ***i***, ***j***, ***k***

  $\boldsymbol{i}^2 = -1$, $\boldsymbol{j}^2 = -1$, $\boldsymbol{k}^2 = -1$,
  $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{jk} = \boldsymbol{i}$, $\boldsymbol{ki} = \boldsymbol{j}$, $\boldsymbol{ji} = -\boldsymbol{k}$, $\boldsymbol{kj} = -\boldsymbol{i}$, $\boldsymbol{ik} = -\boldsymbol{j}$
- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$,   s,x,y,z are scalars

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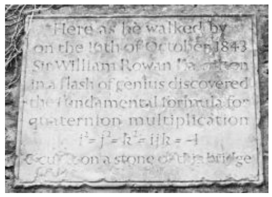
Quaternions

- Invented by Hamilton in 1843 in Dublin, Ireland

- Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication



Source: Wikipedia

$i^2 = j^2 = k^2 = i\,j\,k = -1$

& cut it on a stone of this bridge.

Quaternions

- Quaternions are **not** commutative!

  $q_1 q_2 \neq q_2 q_1$

- However, the following hold:

  $(q_1 q_2) q_3 = q_1 (q_2 q_3)$
  $(q_1 + q_2) q_3 = q_1 q_3 + q_2 q_3$
  $q_1 (q_2 + q_3) = q_1 q_2 + q_1 q_3$
  $\alpha (q_1 + q_2) = \alpha q_1 + \alpha q_2$ (α is scalar)
  $(\alpha q_1) q_2 = \alpha (q_1 q_2) = q_1 (\alpha q_2)$ (α is scalar)

- I.e., all usual manipulations are valid, except cannot reverse multiplication order.

Quaternions

- Exercise: multiply two quaternions

  $(2 - \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})(-1 + \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) = \ldots$

Quaternion Properties

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$
- Norm: $|q|^2 = s^2 + x^2 + y^2 + z^2$
- Conjugate quaternion: $\bar{q} = s - x\,\boldsymbol{i} - y\,\boldsymbol{j} - z\,\boldsymbol{k}$
- Inverse quaternion: $q^{-1} = \bar{q} / |q|^2$
- Unit quaternion: $|q| = 1$
- Inverse of unit quaternion: $q^{-1} = \bar{q}$

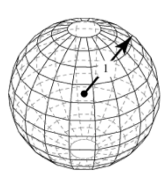
Quaternions and Rotations

- Rotations are represented by ***unit*** quaternions

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

  $s^2 + x^2 + y^2 + z^2 = 1$

- Unit quaternion sphere (unit sphere in 4D)



Source: Wolfram Research

unit sphere in 4D

Rotations to Unit Quaternions

- Let (unit) rotation axis be $[u_x, u_y, u_z]$, and angle θ

- Corresponding quaternion is

  $q = \cos(\theta/2) + \sin(\theta/2)\, u_x\, \boldsymbol{i} + \sin(\theta/2)\, u_y\, \boldsymbol{j} + \sin(\theta/2)\, u_z\, \boldsymbol{k}$

- Composition of rotations q_1 and q_2 equals $q = q_2 q_1$

- 3D rotations do not commute!

Unit Quaternions to Rotations

- Let v be a (3-dim) vector and let q be a unit quaternion
- Then, the corresponding rotation transforms vector v to q ***v*** q^{-1}

  (***v*** is a quaternion with scalar part equaling 0, and vector part equaling v)

For q = a + b ***i*** + c ***j*** + d ***k***

$$R = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Quaternions

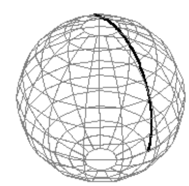
- Quaternions q and -q give the same rotation!
- Other than this, the relationship between rotations and quaternions is unique

Outline

- Rotations
- Quaternions
- Quaternion Interpolation

Quaternion Interpolation

- Better results than Euler angles
- A quaternion is a point on the 4-D unit sphere
- Interpolating rotations corresponds to curves on the 4-D sphere



Source: Wolfram Research

Spherical Linear intERPolation (SLERPing)

- Interpolate along the great circle on the 4-D unit sphere
- Move with constant angular velocity along the great circle between the two points

San Francisco to London

- Any rotation is given by two quaternions, so there are two SLERP choices; pick the shortest

SLERP

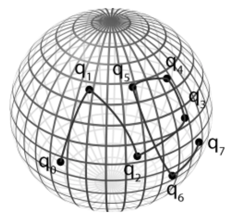
$$\mathrm{Slerp}(q_1, q_2, u) = \frac{\sin((1-u)\theta)}{\sin(\theta)} q_1 + \frac{\sin(u\theta)}{\sin(\theta)} q_2$$

$$\cos(\theta) = q_1 \cdot q_2 =$$
$$= s_1 s_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

- u varies from 0 to 1
- $q_m = s_m + x_m$ ***i*** $+ y_m$ ***j*** $+ z_m$ ***k*** , for m = 1,2
- The above formula does not produce a unit quaternion and must be normalized; replace q by q / |q|

Interpolating more than two rotations

- Simplest approach: connect consecutive quaternions using SLERP
- Continuous rotations
- Angular velocity not smooth at the joints



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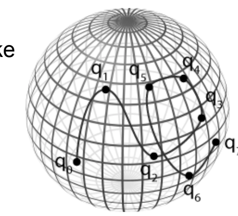
Interpolation with smooth velocities

- Use splines on the unit quaternion sphere
- Reference: Ken Shoemake in the SIGGRAPH '85 proceedings (Computer Graphics, V. 19, No. 3, P. 245)



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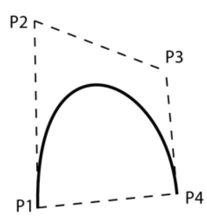
Bezier Spline

- Four control points
  - points P1 and P4 are on the curve
  - points P2 and P3 are off the curve; they give curve tangents at beginning and end



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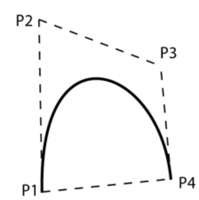
Bezier Spline

- p(0) = P1, p(1) = P4,
- p'(0) = 3(P2-P1)
- p'(1) = 3(P4 - P3)
- Convex Hull property: curve contained within the convex hull of control points
- Scale factor “3” is chosen to make “velocity” approximately constant



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The Bezier Spline Formula

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\text{Bezier basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{Bezier control matrix}}$$

- [x,y,z] is point on spline corresponding to u
- u varies from 0 to 1
- P1 = $[x_1\ y_1\ z_1]$ P2 = $[x_2\ y_2\ z_2]$
- P3 = $[x_3\ y_3\ z_3]$ P4 = $[x_4\ y_4\ z_4]$

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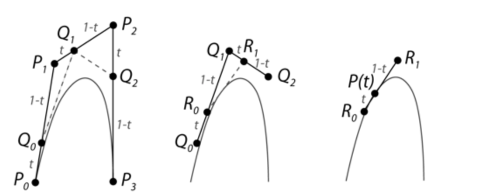
DeCasteljau Construction



Efficient algorithm to evaluate Bezier splines.
Similar to Horner rule for polynomials.
Can be extended to interpolations of 3D rotations.

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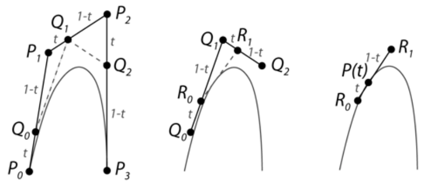
DeCasteljau on Quaternion Sphere

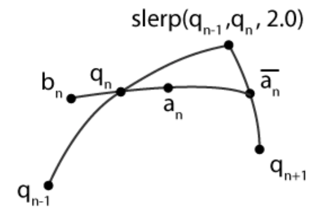


Given t, apply DeCasteljau construction:

Q_0 = Slerp(P_0,P_1,t) Q_1 = Slerp(P_1,P_2,t)
Q_2 = Slerp(P_2,P_3,t) R_0 = Slerp(Q_0,Q_1,t)
R_1 = Slerp(Q_1,Q_2,t)   P(t)= Slerp(R_0,R_1,t)

Bezier Control Points for Quaternions

- Given quaternions q_{n-1}, q_n , q_{n+1} , form:
  $\overline{a_n}$ = Slerp(Slerp(q_{n-1}, q_n, 2.0), q_{n+1}, 0.5)
  a_n = Slerp(q_n , $\overline{a_n}$, 1.0 / 3)
  b_n = Slerp(q_n , $\overline{a_n}$, -1.0 / 3)



Interpolating Many Rotations on Quaternion Sphere

- Given quaternions q_1, ..., q_N ,
  form Bezier spline control points (previous slide)

- Spline 1: q_1, a_1, b_2, q_2
- Spline 2: q_2, a_2, b_3, q_3 etc.

- Need a_1 and b_N; can set
  a_1 = Slerp(q_1, Slerp(q_3, q_2, 2.0), 1.0 / 3)
  b_N = Slerp(q_N, Slerp(q_{N-2}, q_{N-1}, 2.0), 1.0 / 3)

- To evaluate a spline at any t, use DeCasteljau construction