Lagrange mechanics and FEM

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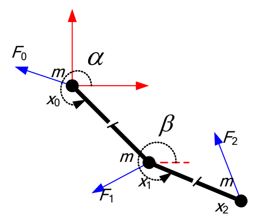
Lagrange mechanics:

As we all know, we have Newton's Law $F = m\ddot{x}$ which describe how the object moves subjecting to external force and we will use this extensively in the assignments. Lagrange Mechanics is extending Newton's Law. For example, we can use it to simulate the robotic arms and human body. Suppose we are given the angles of the joints, we want an equation as:
$q = [\alpha_1, \alpha_2, \cdots, \alpha_n]^T$, $\ddot{q} = F(q)$, this equation is called as Lagrange Mechanics.

The key of Lagrange Mechanics is we have the angles (or any other parameters) which describe the shape of the object. If we know them, we know how the entire structure's position and the motion. Call the function as *G* and the position of the mass function as *x* (2D vector).
$x = [x_1, x_2, \cdots, x_n]^T$, $x = G(q)$, i.e, if we know the angles, we can derive the positions. That's not the differential equation for *x*, but for *q*. When we solve this equation, we get a series of *q*, and then use *G* to solve *x*, and we get the animation.
Why we can not solve the differential equation for *x* directly ($\ddot{x} = F(x)$)? We can do that, but it is another approach, we have constraints, and we get some equations called DAA. There are no constraints if we use α.

If we have kinetic energy T and Potential energy V (gravity, elastic potential energy, etc), we can get Lagrange function $L = T - V$, where $T = T(q, \dot{q})$ and $V = V(q)$, then we have Euler-Lagrange equation



$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \left(\frac{\partial G}{\partial q}\right)^T F_{ext}$$

Concert Example (ACROBOT):
q, *G* function and velocities are:

$$q = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},\ x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ x_1 = \begin{bmatrix} l\cos\alpha \\ l\sin\alpha \end{bmatrix},\ v_1 = \dot{x}_1 = \begin{bmatrix} -l\sin\alpha\cdot\dot{\alpha} \\ l\cos\alpha\cdot\dot{\alpha} \end{bmatrix},\ x_2 = \begin{bmatrix} l\cos\alpha + l\cos\beta \\ l\sin\alpha + l\sin\beta \end{bmatrix}, v_2 = \dot{x}_2 = \begin{bmatrix} -l\sin\alpha\cdot\dot{\alpha} - l\sin\beta\cdot\dot{\beta} \\ l\cos\alpha\cdot\dot{\alpha} + l\cos\beta\cdot\dot{\beta} \end{bmatrix}$$

$$T = \frac{1}{2}m\left(\|v_1\|^2 + \|v_2\|^2\right) = \frac{1}{2}m\left(\dot{\alpha}^2\cdot l^2 + \dot{\alpha}^2\cdot l^2 + \dot{\beta}^2\cdot l^2 + 2l^2\cdot\dot{\alpha}\cdot\dot{\beta}\cdot(\sin\alpha\sin\beta + \cos\alpha\cos\beta)\right) = \frac{1}{2}ml^2\left(2\dot{\alpha}^2 + \dot{\beta}^2 + 2\cos(\alpha-\beta)\dot{\alpha}\cdot\dot{\beta}\right) = \frac{1}{2}\dot{q}^T M\dot{q}$$

Where $\dot{q} = \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$, $M = ml^2\begin{pmatrix} 2 & \cos(\alpha-\beta) \\ \cos(\alpha-\beta) & 1 \end{pmatrix}$ is the mass matrix (function of q) and symmetric.

$V = mg\cdot(2l\sin\alpha + l\sin\beta)$, where $g = 9.81ms^{-2}$, it has no dependence on $\dot{q}$

Now for Euler-Lagrange equation:

$\frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = M\cdot\dot{q}$, thus $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = M\cdot\ddot{q}$

$\frac{\partial L}{\partial q} = \frac{\partial T}{\partial q} - \frac{\partial V}{\partial q}$, where $\frac{\partial T}{\partial q}$ and $\frac{\partial V}{\partial q}$ are as follows:

$$\frac{\partial L}{\partial \alpha} = \frac{1}{2}\dot{q}^T\cdot\frac{\partial M}{\partial \alpha}\cdot\dot{q} - 2lmg\cdot\cos\alpha = \frac{1}{2}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}\cdot\begin{pmatrix} 0 & -\sin(\alpha-\beta) \\ -\sin(\alpha-\beta) & 0 \end{pmatrix}\cdot\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} - 2lmg\cdot\cos\alpha$$

$$\frac{\partial L}{\partial \beta} = \frac{1}{2}\dot{q}^T\cdot\frac{\partial M}{\partial \beta}\cdot\dot{q} - 2lmg\cdot\cos\beta = \frac{1}{2}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}\cdot\begin{pmatrix} 0 & \sin(\alpha-\beta) \\ \sin(\alpha-\beta) & 0 \end{pmatrix}\cdot\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} - 2lmg\cdot\cos\beta$$

For $\frac{\partial G}{\partial q}$, we have $x = \begin{pmatrix} 0 \\ 0 \\ l\cos\alpha \\ l\sin\alpha \\ l\cos\alpha + l\cos\beta \\ l\sin\alpha + l\sin\beta \end{pmatrix}$ $\frac{\partial x}{\partial q} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -l\sin\alpha & 0 \\ l\cos\alpha & 0 \\ -l\sin\alpha & -l\sin\beta \\ l\cos\alpha & l\cos\beta \end{pmatrix}$

For F_{ext} , they are given constant numbers. $F_{ext} = \left(F_{0x}, F_{0y}, F_{1x}, F_{1y}, F_{2x}, F_{2y}\right)$

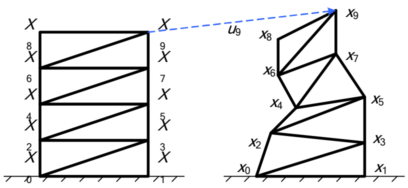
Now we are ready to get the equation: $M\ddot{q} - \overbrace{\begin{pmatrix} \frac{\partial L}{\partial \alpha} \\ \frac{\partial L}{\partial \beta} \end{pmatrix}}^{\alpha,\dot{\alpha},\beta,\dot{\beta}} = \begin{pmatrix} \overbrace{\left(\frac{\partial G}{\partial q}\right)}^{\alpha,\beta} \end{pmatrix}^T \cdot F_{ext}$

FEM

How to relate the Lagrange mechanics to FEM?

For FEM, the principle is the same, but we have different mass matrix, different potential energy etc. Also, we are using continuum mass instead of mass point.

Define q, x, G, T, V

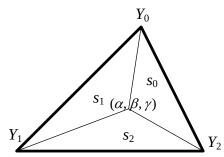


$x_i = X_i + u_i$, $q = \left[u_0, u_1, \cdots u_9\right]^T$,

X (original position) is given, if we know q , we can get x (the new position)

For any point inside the triangle, we use BARYCENTRIC INTERPOLATION.
Considering the point Y in the triangle $Y_0Y_1Y_2$, suppose the total area is s and the subareas are s_0, s_1, s_2, define the coefficients α, β, γ as follows:

$s = s_0 + s_1 + s_2$, $\alpha = \frac{s_0}{s} > 0, \beta = \frac{s_1}{s} > 0, \gamma = \frac{s_2}{s} > 0$, $\alpha + \beta + \gamma = 1$



The point is calculated as: $Y = \alpha Y_0 + \beta Y_1 + \gamma Y_2$

We can write it in a different format as

$$Y = \sum_{i=0}^{2} \psi_i(x) Y_i$$

where $\psi_0(x) = \alpha, \psi_1(x) = \beta, \psi_2(x) = \gamma$, which is called shape function.

Then we get the G function: $x(X) = \sum_{i=0}^{9} \psi_i(X) \cdot u_i$, $G : (q, X) \mapsto x$

Now we are deriving the kinetic energy and potential energy for Euler-Lagrange equation.

Kinetic energy: $T = \frac{1}{2}\dot{q}^T M \dot{q}$, which is similar as before but M (20*20 matrix) is constant and depend on the geometry of the object.

Potential energy (here we only consider elastic potential). In order to derive it, we need to figure out how much each tiny little piece is deformed, then do an integral of the entire continuum to get the entire energy (elastic potential or elastic strain energy) as follows:

$V = \int_{\Omega} v(X) ds$, where $v(X)$ is the density of the energy.

Let's derive the $v(X)$. In order to do that, we have to define the mapping Φ from the original domain X to the deformed domain x . $\Phi(X) = \sum_{i=0}^{9} \psi_i(X) \cdot u_i$

Deformation gradient (3x3 matrix in 3D): $F = \frac{\partial \Phi}{\partial X}$

Then there is something called strain tensor (3x3 matrix in 3D): $E = \frac{1}{2}\left(F^T F - I\right)$

In addition, we have to define some function which map E into S (Stress tensor): $S = \Psi\left(E\right)$, it is extension of Hook's Law as we used in the previous lectures.

Then we can define $v(X) = S : E$ (: means pair-wise multiplication).

Now we get the equation we talked last time:

The mapping Ψ could be common used linear material:

$S = \lambda \cdot \mathrm{tr}\left(E\right) \cdot I + 2\mu E$