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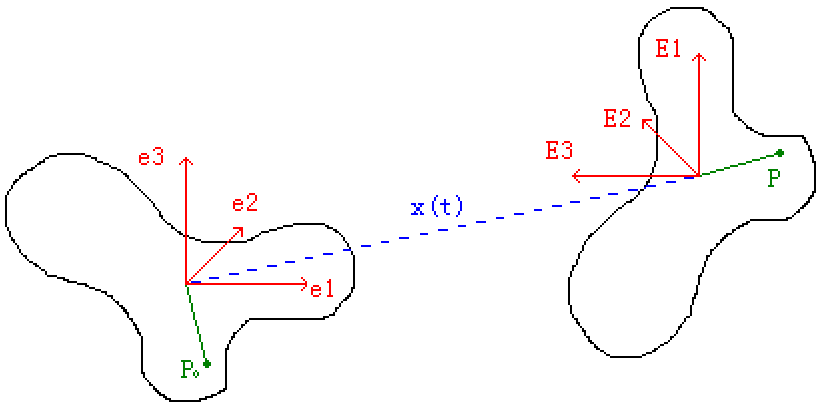
Rigid body dynamics

Basic representation

We generalize the concept of motion of a body by defining a state vector:

$X(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$, it can be described as an array of 6 numbers.

The change of X(t) over time is given by: $\dot{X}(t) = \begin{bmatrix} v(t) \\ F(t)/m \end{bmatrix}$



The change of object state from the left to right through translation matrix X(t) and rotation matrix R(t), then we have: $R(t) \cdot e_i = E_i (i = 1,2,3)$ (Note: R(t) is a 3x3 orthogonal matrix, det R = 1)

Therefore: $R(t) = [E_1 E_2 E_3]$

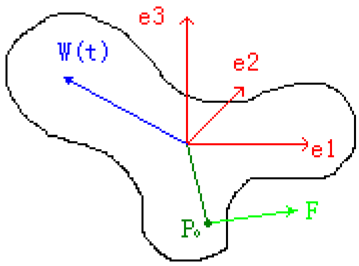
Center of mass = $\frac{1}{m} \int_{\Omega} \vec{r} \cdot \rho \cdot dv$

$P(t) = X(t) + R(t) \cdot P_0$

$\dot{P} = \dot{X} + \dot{R} \cdot P_0$

Angular Velocity

In addition to translation, a rigid body can also rotate. The rotation of the body must about some axis passes through the center of the mass. We describe the rotation as a vector $\omega(t)$



$\tau = (P(t) - X(t)) \times F\,;\;\; y \mapsto a \times y$

Particularly, given a vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix};\;\; a \times y = \tilde{a} \cdot y$

$$\tilde{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$\tilde{a} = -\tilde{a}^T$ (Skew-Symmetric)

We do the similar thing to ω:

$$V = \omega \times P = \omega \times R \cdot P_0 = \dot{R} \cdot P_0$$

$$\dot{R} \cdot P_0 = \tilde{\omega} \cdot (RP_0) = \tilde{\omega} R P_0$$

$$\dot{R} = \tilde{\omega} R$$

$$\tilde{\omega} = \dot{R} R^T$$

Now the representation changes to:

$$X(t) = \begin{bmatrix} x(t) \\ v(t) \\ R(t) \\ \omega(t) \end{bmatrix}$$

Inertia tension

$$I(t) = \int \rho \cdot (\|F\|^2 \cdot I - F^T F) dv$$

I(t) is a 3x3 matrix (a rank 2 tensor)

For example:

The inertia tension of a concrete sphere is: $I(t) = \begin{bmatrix} \frac{2}{5}mR^2 & 0 & 0 \\ 0 & \frac{2}{5}mR^2 & 0 \\ 0 & 0 & \frac{2}{5}mR^2 \end{bmatrix}$

Linear Momentum

The linear momentum p of a particle with mass m and velocity v is defined as:

$$P = m \cdot v(t)$$

The conservation of linear momentum is:

$$\int F(t)dt = \Delta P$$

Therefore, we have: $\dot{P} = F(t)$

Angular Momentum

Usually we use angular momentum to simplify our equations. It's conserved by nature.

$L(t) = I(t) \cdot \omega(t)$ where $I(t) = R(t) \cdot I_{body} \cdot R(t)^T$

Conservation law:

$$\int \tau(t)dt = \Delta L$$

$$\dot{L} = \tau(t)$$

The representation now is:

$$X(t) = \begin{bmatrix} x(t) \\ P(t) \\ R(t) \\ L(t) \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} P/m \\ F(t) \\ \tilde{\omega}R \\ \tau(t) \end{bmatrix}; \quad \omega = I(t)^{-1} L(t)$$