

## Structured Deformable Objects: FEM

### §Part 1. Interactive Virtual Material

FEM was mostly studied by mathematicians in the 60's. It was firstly employed by *Müller and Gross* (2004) to successfully simulate 3D solid using computer graphics. After that, FEM method has been an active research area in computer graphics. According to *Müller and Gross's* work [1] covered several aspects in solid simulation:

- Linear methods: Good for small deformation (80% simulations in engineering concern about small deformations, such as earthquakes, bridge etc.). However it is not accurate for large deformation, as shown in Figure 1 [1]. Some modification has been made and will be discussed in part 2.
- Plasticity: Plasticity corresponds to non-elastic deformation of an object.
- Fracture: To simulate something complex, more details are needed but they will slow down the simulation. One can build a coarse mesh to simulate the object and interpolate the fine model into the coarse mesh. In this paper the authors build tetrahedron meshes on one surface and simulate the fracture on the coarse mesh and then re-mesh the part that has been exposed.



Figure 1. The left figure shows the artifact of linear FEM under large rotational deformation. The correct deformation is shown on the right.

Recall that in the home assignment 1 we employed mass-spring model to simulate a jello-cube. It can also be a good model to simulate cloth, but in 3D object simulation FEM is better.

FEM vs Mass-spring system:

FEM: Good for volume control.

Mass-spring: It does not preserve material's volume.

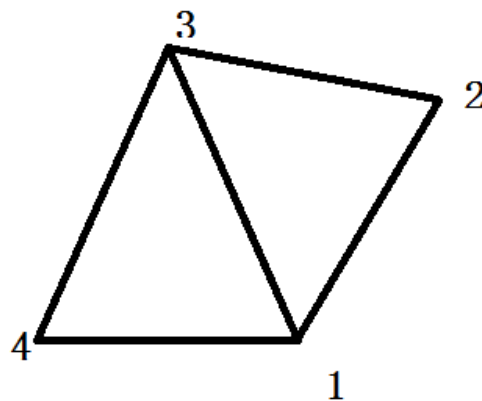
One can create a computer graphics object by putting a real object into a laser scanner. It will build the mesh triangles automatically; Otherwise one can buy a mesh online or hire a computer graphics artist.

.Obj mesh format file example: (Widely adopted method for storing computer graphics data by MAYA, industry, academia)

```
v    0.25  -0.3  0.8  // Vertex 1
v    0.5   0.2  0.75 // Vertex 2
v    ...   ...   ...
...  ...   ...   ...
f    1     3     4
f    1     2     3
```

(or if it is a quadrant)

```
f    1     2     3     4
```



Software will process the mesh by drawing facets according to f (facet) and v (vertex).

Dead mesh can be used to preserve the volume.

## §Part 2. Finite Element Method (FEM)

In general to simulate a 3D solid object deformation one has to solve partial differential equations (PDEs). By using the finite element method, the problem of solving PDEs can be simplified to just solving easier ordinary differential equations (ODEs) with some cost of accuracy. Here is an example of FEM problem. Figure 2 shows a deformed object whose upper-right corner has been moved from position  $P$  to  $P'$ .

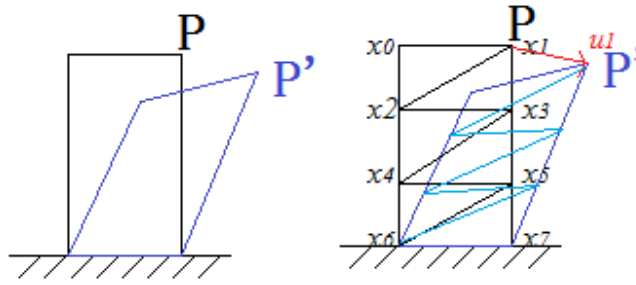


Figure 2. A 2D deformed object as an FEM example. The object is subdivided into 8 parts on the right.

To precisely model the deformation  $u(x, y)$  on every point of this object, one has to solve the PDEs below:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & (\text{volume constraint}) \\ u(P) = d & (\text{known displacement of } P). \\ \frac{\partial u}{\partial n} |_{\partial\Omega} = 0 & (\text{boundary condition}) \end{cases}$$

However, to solve this PDE is computationally costly. FEM method can be employed to simplify the computation with some accuracy tradeoff. By dividing the object into 8 parts, one can build a linear function to solve  $u_0, u_1, \dots, u_7 : \mathbf{A} \cdot \mathbf{u} = \mathbf{b}$  where  $\mathbf{u} = (u_0^T, u_1^T, \dots, u_7^T)^T$ , and interpolate the displacements of other points.

In [1], the non-linear PDEs for elasticity was transformed into ODE by FEM, which has the following form:

$$M\ddot{\mathbf{u}} + D(\mathbf{u})\dot{\mathbf{u}} + f_{int}(\mathbf{u}) = f_{ext}(t).$$

The interior force term  $f_{int}(\mathbf{u})$  is further approximated by  $K\mathbf{u}$  where  $K$  is a coefficient related to the Young's Modulus.

To address the deformation with rotation, the authors compute the elastic forces in 3 steps, as shown in figure 3[1]:

- (1) Compute the best rotational matrix  $R_e$  of a tetrahedron (by polar decomposition);
- (2) Compute the force using linear approximation:  $K_e(R_e^{-1}\mathbf{x} - \mathbf{x}_0)$ ;
- (3) Rotate the force back:  $R_e K_e(R_e^{-1}\mathbf{x} - \mathbf{x}_0)$ .

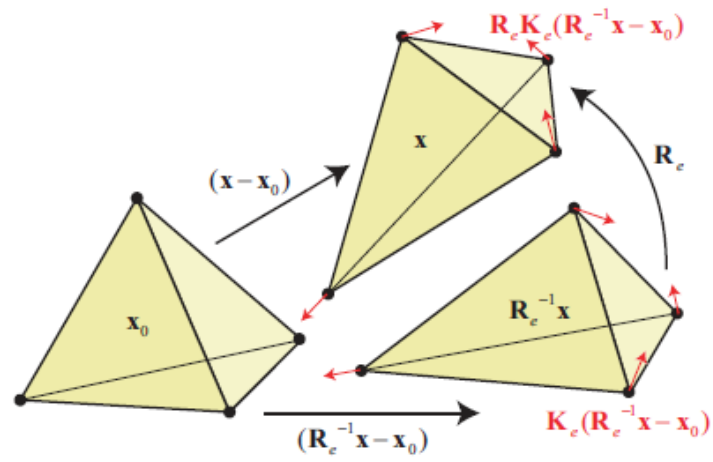


Figure 3. Elastic force computation step.

### Reference:

- [1] M. Müller and M. Gross: [Interactive virtual materials](#), Graphics Interface 2004.