Real-Time Subspace Integration for St.Venant-Kirchhoff Deformable Models

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Author: Samuel Yung

1. Introduction

Real Time Subspace Integration for St Venant Kirchhoff (St VK) Deformable Models is a fast method to simulate Finite Element Method (FEM) deformable objects. This fast method can interactively simulate non-linear large deformation.

2. Background

2.1 Model Reduction

Huge high dimensional system, such as fluids and deformable objects, have n degrees of freedom, n vertices, and very slow to time step. These complex models can be very computationally expensive. How can you simulate these models efficiently? One method is to use model reduction, which reduces a huge high dimensional system into a simpler high dimensional system that will be used to approximate the complex model.

2.2 Solid Mechanics Internal Forces under Deformation

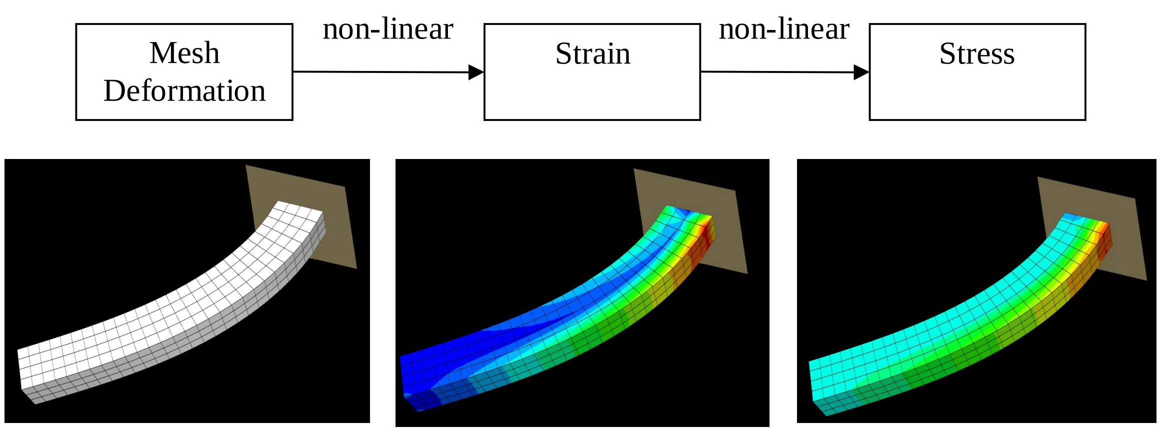


Figure 1. The standard pipeline internal forces in deformation.

When an object is deformed there are internal forces that return objects to the original shape. To compute the deformation we have to compute strain, which describes how an object is deformed from the mesh deformation. However, computing strain is not enough, since it is a geometric measure and does not capture any internal forces or material properties. Thus we use Material Law relationship to compute stress from strain, which calculates the internal forces incorporating the object's material property. By combining all these together we have the standard pipeline for internal forces in model deformations.

2.3 St. Venant-Kirchhoff Deformable Model

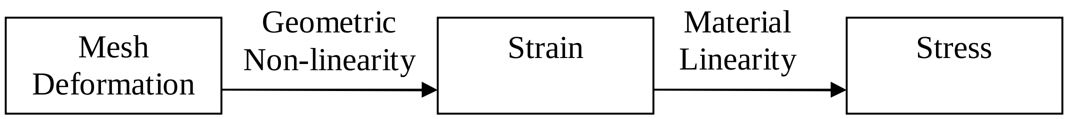
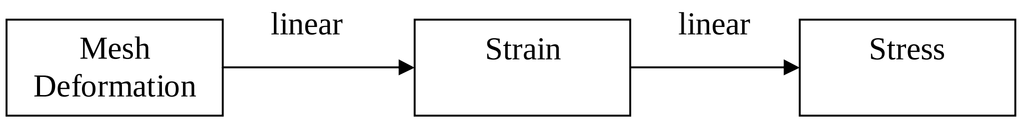


Figure 2. St Venant-Kirchhoff Deformable Model Definiton.

In the St Venant Kirchhoff Deformable Model, the pipeline is similar to the solid mechanics internal forces deformation model. However, mesh deformation has a geometric non-linearity relationship with strain, while strain has a material linearity relationship with stress.

2.4 Linear Modal Analysis



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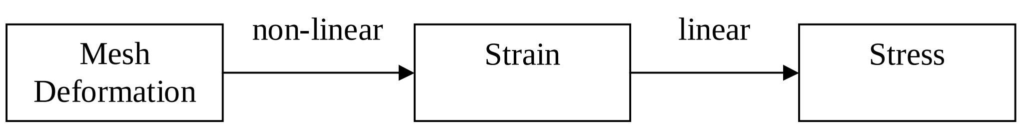
Simulate in a low-frequency deformation basis

Figure 3. Linear Modal Analysis Definition.

The Linear Modal Analysis is applying reduction to the simplest deformation model. The general idea is to extract natural frequencies of objects and combine it with the simplest deformation model, which has a geometric linearity and material linearity. The benefits of Linear Modal Analysis are the speed and its usage of the GPU. Unfortunately, the down fall with this method is Linear Modal Analysis can only handle purely linear deformations.

3. Subspace Integration

3.1 Subspace Integration Deformable Model



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Simulate in a quality low-dimensional basis of
non-linear deformations.

Figure 4. Subspace Integration Deformable Model Definition.

Real-Time Subspace Integration expands the idea of Linear Modal Analysis, but utilizes geometric non-linearity instead of geometric linearity. In addition, this method combines the deformation model with a low-dimensional basis. Simulating in low quality space obtains quality deformations. As a result, we have faster non-linear dynamics and remove artifacts of linear modal analysis.

3.2 Subspace Integration

To compute how the objects' deformation evolves through time, we utilize 3D Continuum Mechanics and FEM equations of motion:

$$M\ddot{u}(t) + R(u(t)) = f(t), \quad where: u = \underset{3n\,x\,1}{\Big|} \; (Deformation\,Vector) \tag{1}$$

u – Deformation Vector that contains the 3D deformation vectors for all mesh vertices.
M – Mass matrix of the object (Sparse Matrix).
$M\ddot{u}(t)$ – Acceleration of object.
$R(u(t))$ – Internal forces to deform the object back.
NOTE: Rest state for the object is when u = 0.

To simulate deformation evolution over time in Subspace with model reduction, we restrict the deformation vector *u* in 4D space with linear combinations of key basis shapes. Through model reduction, this creates a displacement basis matrix $U \in \mathrm{R}^{3n,r}$ and a vector of reduced coordinates, $q \in \mathrm{R}^{3n}$. In addition, *U* is a constant and a time independent matrix specifying a basis of some *r* - dimensional linear subspace of R^{3n}. By multiplying *U* and *q* we acquire an approximation of the High Dimensional System, *u*. Thus we replace *u* with *Uq*.

$$M\ddot{u} + R(u) = f, \quad where: u = Uq \tag{2}$$

By simply replacing *u* with *Uq* we obtain too many rows for computation. To reduce the number of rows, we need to pick a specific displacement basis for this subspace. How do you get the *U* basis? We must have a basis that captures typical non-linear deformations. For this subspace, the specific deformation basis is an orthogonal basis or make the columns of *U* mass-orthogonal. To accomplish this we need to project. In the projection step, we pre-multiply by U^T.

$$M\ddot{u} + R(u) = f \tag{3}$$

$$\Big\Downarrow \begin{array}{l} \textit{Substitute } u = Uq \\ \text{Project } U^T \end{array}$$

$$\overbrace{U^T M U}^{Identity}\ddot{q} + U^T R(Uq) = U^T f \tag{4}$$

$$\Downarrow$$

$$\underbrace{\ddot{q}(t) = \tilde{R}(q(t)) = \tilde{f}(t)}_{\text{Reduced Equations of Motion}} \tag{5}$$

After inserting $u = U^T$ and pre-multiplying by U^T, we derive a Reduced Equation of Motion:

$$\ddot{q}(t) = \tilde{R}(q(t)) = \tilde{f}(t) \tag{6}$$

$$\tilde{R}(q) = U^T R(Uq) = \underbrace{\begin{pmatrix} p_1(q) \\ p_2(q) \\ \ldots \\ p_r(q) \end{pmatrix}}_{r \textit{ cubic polynomials in component of } q} \tag{7}$$

In the Reduced Equation of Motion, $\tilde{R}(q)$ become the reduced internal forces of the object. Additionally, each component of $\tilde{R}(q)$ is a multivariate cubic polynomial in components of reduced coordinates q. Due to this outcome, the cubic polynomial coefficients are constants. Furthermore, we can compute the gradient of the matrix, $\tilde{R}(q)$, for increasing integration:

$$\frac{\partial \tilde{R}(q)}{\partial q} = \underbrace{\begin{pmatrix} s_{11}(q) & \dots & s_{1r}(q) \\ \dots & \dots & \dots \\ \\ s_{r1}(q) & \dots & s_{rr}(q) \end{pmatrix}}_{r^2 \ quadratic \ polynomials \ in \ q} \tag{8}$$

The result is a quadratic polynomial, which turns out to be the Reduced Stiffness Matrix. Since the polynomials are low degree, we can efficiently pre-compute these coefficients and the gradient of the matrix, $\tilde{R}(q)$. In comparison, we have the Unreduced Equation of Motion, which uses equation (1), high dimension (e.g. 3n = 75000) and needs $R(u)$, $\frac{\partial R(u)}{\partial u}$; and the Reduced Equation of Motion uses equations (6), low dimension (e.g. r = 30), and needs $\tilde{R}(q)$, $\frac{\partial \tilde{R}(q)}{\partial q}$. IN conclusion, Subspace Integration can interactively simulate non-linear large deformation.

3.3 Subspace Integration Limitations

There are several limitations of Subspace Integration. One limitation that is the polynomial evaluations takes $O(r^4)$ time. Another limitation are local deformations may need large basis (Multi-resolution extensions). Lastly, St VK is inaccurate for large compressions in non-linear materials; this includes reduction, sketch, and modal derivatives.