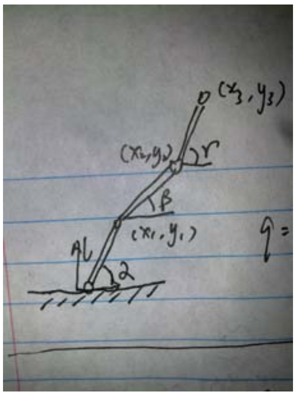
CS 599 Physically Based Scribe Notes

Wednesday 2/16/11 Scribe – Yufeng Zhu

Hinge joint example:



$q=[\alpha, \beta, \gamma]$, (Minimal Coordinates) $M(q)*q''=f(q,q'',t)$

In our case, we use Maximal coordinates which looks like $q=[x_1,y_1,x_2,y_2,x_3,y_3]$. (6 DOF)
The constraint equation is as follows:

$$x_1^2+y_1^2-l^2=0$$
$$(x_2-x_1)^2+(y_2-y_1)^2-l^2=0$$
$$(x_3-x_2)^2+(y_3-y_2)^2-l^2=0$$

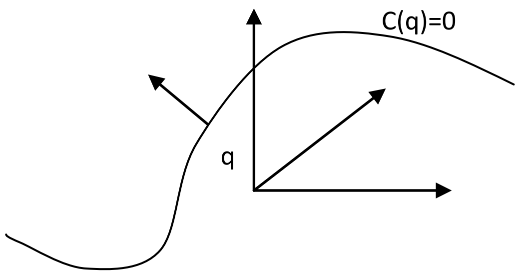
(3 constraints)

To keep the mathematical convention form, we would like to represent the above constraint equations in a vector C(q) which we call constraint vector.

$$C(q)=[\, x_1^2+y_1^2-l^2,\ (x_2-x_1)^2+(y_2-y_1)^2-l^2,\ (x_3-x_2)^2+(y_3-y_2)^2-l^2]=0^3$$

Constraint visualization:

If q is in N dimension space, C(q)=0 is kind of hyper iso surface in the space.



More General Case
Assume the DOF is n and constraint is m. Then we have two vectors.

$$C=[C_1, C_2,\cdots C_m]$$

$q=[q_1, q_2,\cdots q_n]$

Then we can calculate the constraint gradient which is a mxn Jaccobian Matrix.

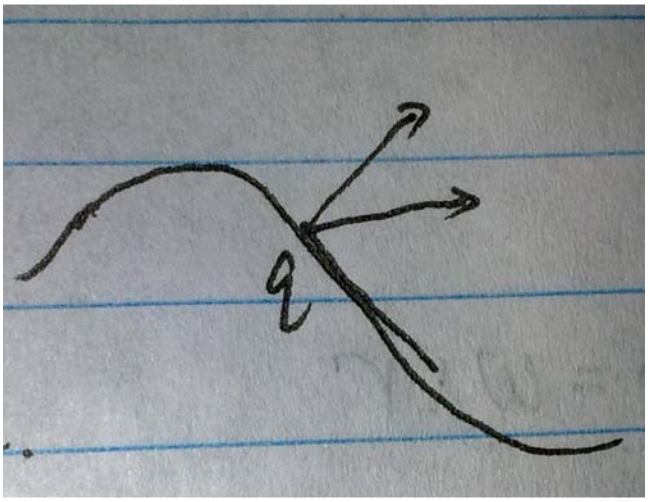
$$\frac{dC}{dq}=\begin{bmatrix} \frac{dC1}{dq1} & \cdots & \frac{dC1}{dqn} \\ \vdots & \ddots & \vdots \\ \frac{dCm}{dq1} & \cdots & \frac{dCm}{dqn} \end{bmatrix}$$

Then we can get the constraint force.

$$F_c=(\frac{dC}{dq})^T * \lambda, \qquad \lambda=[\lambda_1, \lambda_2,\cdots \lambda_m] \text{ (Lagrange Multiplier)}$$

Now we come to our governing equations:

$m_i*x''_i=f_i^{ext}+f_i^{constrain}$ (net energy should stay the same)



Combining with the constraint condition, we have

$$\begin{cases} M*q''=\mathrm{F}-\mathrm{Grad(C)}*\lambda \\ C(q)=0 \end{cases},$$

Grad(C) is the transpose of the C's gradient and F is f^{ext}.

Since $C(q)\equiv 0$, $\frac{dC(q(t))}{dt}=0$

Using the chain rule, we have $\frac{dC}{dq}*\frac{dq}{dt}=0$. Then we take second derivative over this equation. We get $\frac{dC}{dq}*q''+\left(\frac{d}{dt}\left(\frac{dC}{dq}\right)\right)*\frac{dq}{dt}=0$.

Using the product rule, namely $\frac{d}{dt}\left(\frac{dCi}{dqj}\right)=\frac{d}{dqj}\left(\frac{dCi}{dt}\right)=\frac{d}{dqj}C_i'$, we have $\frac{dC}{dq}*q''+\left(\frac{d}{dq}C'\right)*\frac{dq}{dt}=0$.

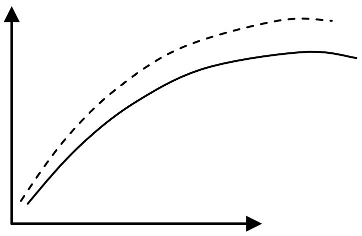
Now our governing equations can be written like this:

$$\begin{cases} M * q'' + \text{Grad(C)} * \lambda = \text{F} \\ \frac{dC}{dq} * q'' = -\left(\frac{d}{dq} C'\right) * \frac{dq}{dt} \end{cases}$$

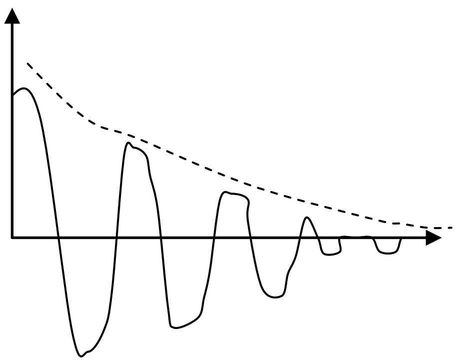
Again all we need to do is to comply our mathematical convention, representing in matrix form.

$$\begin{bmatrix} M & \text{Grad(C)} \\ \frac{dC}{dq} & 0 \end{bmatrix} * \begin{bmatrix} q'' \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ -\left(\frac{d}{dq} C'\right) * \frac{dq}{dt} \end{bmatrix}$$

However, this is not the end. Since we are solving the system using numerical solution, we cannot numerical error. Even for symbolic solution, common computer calculation will introduce error to the result, which makes the state leave the correct state. This will lead the solution diverging finally as shown in the following figure.



To overcome this problem, we utilize a technique called Baumgarte Stabilization which is to replace C''=0 with C''+α*C'+β*C=0 (harmonic oscillator). In this case, even we may have some numerical error which causes C≠0 during the solution, but it will finally converge to zero. The process can be visualized as follows.



Thus the final governing equations are

$$\begin{bmatrix} M & \text{Grad(C)} \\ \frac{dC}{dq} & 0 \end{bmatrix} * \begin{bmatrix} q'' \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ -\left(\frac{d}{dq} C'\right) * \frac{dq}{dt} - \alpha * \frac{dC}{dq} * q' - \beta * C \end{bmatrix}$$

But how to choose α and β still needs manual tweaking.