On the M^{-1} norm and Equation 2 in [Barbič et al. 2012]

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Supplementary material to "Interactive Editing of Deformable Simulations"

Abstract

This document derives Equation 2 in [Barbič et al. 2012].

1 Derivation of Equation 2

First, let us explain why we used the M^{-1} weighted energy in Equation 1 in [Barbič et al. 2012]. Such an "energy" measures the total sum of squares of impulses (momentum; force times timestep) applied at each timestep (defined as Fh, where h is timestep). In mechanics, the proper norm for momentum is the inverse of mass. For a particle with momentum p, such a norm gives the particle's kinetic energy:

$$W_{\mathrm{kin}} = \frac{1}{2} < \frac{1}{m} p, p >= \frac{1}{2} < v, mv > . \tag{1}$$

Similarly, the proper norm for v is simply the mass, because it gives the kinetic energy: $1/2 < mv, v >$.

The M^{-1} norm has the very elegant property that it works well with model reduction (which is usually a sign that the norm is the correct one). When inserting $p = Uz$ into Equation 1 of [Barbič et al. 2012], with the M^{-1} weighting, the expression for E can be simplified as follows. Here, we omit the constant factors and the temporal discretization for brevity.

$$E = ||Mp'' + Dp' + Kp||^2_{M^{-1}} = ||MUz'' + DUz' + KUz||^2_{M^{-1}} = \tag{2}$$

(Rayleigh damping)

$$= ||MUz'' + (\alpha M + \beta K)Uz' + KUz||^2_{M^{-1}} = \tag{3}$$

(U are the modes: $KU = MU\Lambda$)

$$= ||MU(z'' + \alpha z') + MU\Lambda(z + \beta z')||^2_{M^{-1}} = \tag{4}$$

(collecting MU)

$$= ||MU(z'' + \alpha z' + \Lambda(z + \beta z'))||^2_{M^{-1}} = \tag{5}$$

(definition of M^{-1} norm)

$$< M^{-1}MU(z'' + \alpha z' + \Lambda(z + \beta z')), MU(z'' + \alpha z' + \Lambda(z + \beta z')) >= \tag{6}$$

(cancel M^{-1} and M)

$$< U(z'' + \alpha z' + \Lambda(z + \beta z')), MU(z'' + \alpha z' + \Lambda(z + \beta z')) >= \tag{7}$$

(move U in first factor to the other side (U^T))

$$< z'' + \alpha z' + \Lambda(z + \beta z'), U^T MU(z'' + \alpha z' + \Lambda(z + \beta z')) >= \tag{8}$$

(modes are mass-orthonormal, $U^T MU = I$)

$$< z'' + \alpha z' + \Lambda(z + \beta z'), z'' + \alpha z' + \Lambda(z + \beta z') >= \tag{9}$$

(definition of 2-norm)

$$= ||z'' + \alpha z' + \Lambda(z + \beta z')||^2_2 = \tag{10}$$

(rearrange terms to collect damping terms)

$$= ||z'' + (\alpha + \beta\Lambda)z' + \Lambda z||^2_2, \tag{11}$$

which is the quantity used in Equation 2 in [Barbič et al. 2012].

References

BARBIČ, J., SIN, F., AND GRINSPUN, E. 2012. Interactive editing of deformable simulations. *ACM Trans. on Graphics (SIGGRAPH 2012) 31*, 4.