CS170: Discrete Methods in Computer Science
Summer 2023
Advanced Induction

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[1] These slides adapt some content from similar slides by Aaron Cote.

Outline

1. Strengthening the Inductive Hypothesis

2. Inductive Definitions and Structural Induction

Strengthening the Inductive Hypothesis

- Recall: In induction, you want to prove some property $p(n)$ for all $n \geq n_0$
- Typically, you approach this as follows:
  - You prove p for one or more base cases
  - You assume that p holds for arbitrary n (or up to n in strong induction), and then prove it for $n + 1$.
- Sometimes this doesnt work
  - The inductive hypothesis p for n (or up to n) does not actually imply $p(n + 1)$
- But something more clever does!

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Strengthening the inductive hypothesis

A stronger claim $q(n)$ might work! Even though it is stronger ($q(n) \Rightarrow p(n)$), so feels like it should be harder to prove, it can give you the strength you need to prove the inductive step $q(n + 1)$.

Example

Claim

$$\sum_{i=1}^{n} \frac{1}{2^i} < 1$$

On Board

Example

Claim

A $2^n \times 2^n$ chessboard with one of the four middle squares removed can be tiled by 3-square L-shaped pieces.

On Board

Example

Claim

$\sum_{i=1}^{n}(2i-1)$ is a perfect square for all n.

On Board

Outline

Inductive (a.k.a. Recursive) Definitions

- Sometimes we define a set S recursively by
  - Base cases: Defining its "smallest" members
  - Inductive (a.k.a. Recursive or Constructor) cases: Describing how to construct more and more members of S from existing ones.
- This is called an inductive or recursive definition.
- Implicit in such a definition is the stipulation that members of the set are exactly the base cases and anything that can be constructed from them by applying the inductive cases recursively (a finite number of times).

Example: Linked List

In a functional programming language, you might define a linked list as follows:

Linked List

A linked list of objects in E is either

- The empty list, denoted by $()$, or
- (x, L) where $x \in E$ and L is a linked list of objects in E

E.g. $E = \mathbb{N}$

Example: Propositional Logic Formulae

A formula in propositional logic is either

- A propositional variable ($p, q, r, \ldots$), or
- $\neg A$ where A is a formula, or
- $A \wedge B$ where A and B are formulas, or
- $A \vee B$ where A and B are formulas, or
- $A \Rightarrow B$ where A and B are formulas

Example: String of Balanced Parentheses

A string of balanced parentheses is either

- The empty string, or
- (S) where S is a string of balanced parentheses, or
- $S_1 S_2$ where S_1 and S_2 are strings of balanced parentheses

Structural Induction

- Sometimes, you need to prove a property of objects in a set that is defined inductively
- Given a suitable inductive hypothesis (the property itself, or something stronger), it suffices to:
  - Prove it for the base cases
  - For each of the recursive cases, prove that if it holds for the inputs then it holds for the output
- This is called structural induction, since it follows the recursive structure of the definition

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Note

This is not more general than regular induction. It is equivalent to inducting on the number of recursive calls needed to construct the object. However, it is a convenient abstraction to work with.

Example

Claim

A string of balanced parentheses has the same number of (and).

On Board

Example

Claim

A propositional logic formula can be re-written using only the operators $\wedge$ and $\neg$.

On Board