CS170: Discrete Methods in Computer Science
Summer 2023
First Order Logic

Instructor: Shaddin Dughmi[1]

[1] These slides adapt some content from similar slides by Aaron Cote.

Outline

Introduction

- In propositional logic, atomic statements were simply propositional variables which may be true or false
  - We built formulas from them using operators like $\neg, \wedge, \vee, \Rightarrow$.
- More generally in math, we want to make statements about a universe of objects
  - E.g. Integers, USC students, graphs, sets, algorithms
- For this, we need statements that take arguments. These are predicates.
- The arguments can be constants in the universe, or variables, or more generally a term (more on this later)
  - E.g. $P(1)$, $P(x)$, $P(x+y)$, $P(sqrt(x))$ if the universe is the real numbers
- From these predicates, we can construct formulas using quantifiers $\forall$ and $\exists$, in addition to the usual language of propositional logic

Predicates

- A predicate is a statement that takes in zero or more (variable) arguments
  - We usually denote predicates by symbols like P, Q, . . . , followed by arguments in parentheses
  - Some predicates with special meaning can be given special symbols (e.g. $=, >, <, \ldots$)
- Examples of predicates and associated universes
  - $P(n)$ = "n is odd", over the universe of integers.
  - $P(x)$ = "x lives off campus", over the universe of USC studnets
  - $P(m, n)$= "m divides n", over the universe of integers
- A predicate becomes a proposition when its arguments are replaced by constants in the universe

Universe and Terms

- In first order logic, there is a universe $\mathbb{U}$ and functions on it
  - E.g. $\mathbb{U} = \mathbb{Z}$
  - There are infinitely many generic functions $f, g, \ldots$
  - Also special functions relevant to your universe like $+, -, \times, \ldots$
- A term stands for something in the universe, such as
  - A variable x, y, z, $\ldots$
  - A constant, which can be
    - a generic symbol like a,b,c,$\ldots$
    - a special symbol like 0,1, -7 for constants you know something about
  - An expression involving variables, constants, and functions, such as $x \times y - 7 \times a$, or $f((x + 1) \times g(b))$.
- A predicate is allowed to take in a term for each of its arguments
  - E.g. $P(x \times y - 7 \times a, f((x + 1) \times g(b)))$ for a 2-argument predicate on the integers.

Building Formulas

A first-order formula is either

- A predicate with terms as arguments (Base case)
- A combination of other formulas using propositional operators $\wedge, \vee, \neg, \Rightarrow$ (Recursive case 1)
- $\forall x F$ or $\exists x F$ for a formula F (Recursive case 2)

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Note

- We use parentheses for clarity as usual
- It is common to always allow the special predicate $=$ on the universe (makes life easier)
- A variable in a formula may be free or bound

Free and Bound Variables

- An occurence of a variable x in a formula can be free or bound
- Bound: There is a quantifier that tells us whether we should be checking the formula for all x or only for some x
- Free: There is no such quantifier, so it's a “loose reference”
- A formula with no free variables has a definite truth value (whether you know it or not), otherwise it does not

Outline

Worked out on board

Outline

Inference Rules of First-Order Logic

- Inference rules of propositional logic: (see corresponding lecture)
- Quantifier Negation:

$$\neg\forall x F(x) \equiv \exists x \neg F(x)$$

$$\neg\exists x F(x) \equiv \forall x \neg F(x)$$

- Change of Variables:

$$\exists x F(x) \equiv \exists y F(y)$$

if y is a fresh (new) variable.

- Scope change:

$$\exists x(F(x) \vee G) \equiv (\exists x F(x)) \vee G$$

$$\forall x(F(x) \vee G) \equiv (\forall x F(x)) \vee G$$

$$\exists x(F(x) \wedge G) \equiv (\exists x F(x)) \wedge G$$

$$\forall x(F(x) \wedge G) \equiv (\forall x F(x)) \wedge G$$

if G does not mention x.

Inference Rules of First-Order Logic

- Universal Instantiation:

$$\forall x F(x) \vdash F(t)$$

  for any term t.
- Existential Instantiation:

$$\exists x F(x) \vdash F(a)$$

  for some constant a. Must be a fresh (new) symbol.
- Universal Generalization:

$$F(a) \text{ for arbitrary constant } a \vdash \forall x F(x)$$

  a must not have previous mention in assumptions or proof.
- Existential Generalization:

$$F(a) \vdash \exists x F(x)$$

  No restriction on the constant a

Soundness and Completeness

Recall the following two properties of a logic with rules of inference.

- Soundness: If there is a proof that starts with a set of assumptions and derives a conclusion, then the conclusion is logically entailed by the assumptions.
  - Informally: "Everything you prove is in fact true."
- Completeness: If a conclusion is logically entailed by a set of assumptions, then there is a proof that starts with those assumptions and derives the conclusion.
  - Informally: "Everything true has a proof."

Here, logically entailed means there's no way to construct predicates that satisfy assumptions without satisfying conclusion.

Soundness and Completeness

Just like in propositional logic, we have

Theorem

First-Order Logic, with the inference rules I showed you, is both sound and complete.

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Theorem

First-Order Logic, with the inference rules I showed you, is both sound and complete.

Note

Not to be confused with the Incompleteness Theorems (which you may or may not have heard of). Those are different!