CS170: Discrete Methods in Computer Science
Summer 2023
Mathematical Induction

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[1] These slides adapt some content from similar slides by Aaron Cote.

Outline

Intuition

Captures the following types of scenarios:

- You have a row of dominos that goes off into the horizon. You knock over the first domino. Each domino knocks over the next one. We can conclude that all dominoes get knocked over.

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- You prove a predicate for the number 1. Then you prove that if the predicate holds for i, then it holds for $i + 1$. We can conclude that the predicate holds for all positive integers.

Inuition

- At a high level, it allows you to prove something about bigger and bigger integers one step at a time ("inductively").
- Though it is technically about integers, can be used to prove general statements about all sorts of mathematical objects like sets, functions, graphs, games, algorithms, etc.
  - This is because you can parametrize size of the object by an integer, then use induction to prove for an object of arbitrary size.

Example: Sum of Consecutive Integers

Claim

The sum of the first n positive integers is $\frac{n(n+1)}{2}$. In mathematical notation:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

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$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \text{ (by the inductive hypothesis)} \\
&= (n+1)(\frac{n}{2} + 1) \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

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which is exactly IH($n+1$).

Example: Telescoping Product

Claim

The product of $1 + \frac{1}{i}$ for i from 1 to n is $n + 1$. In mathematical notation:

$$\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = n + 1$$

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- Induction Step: Assume IH(n) and prove IH($n + 1$):

$$\begin{aligned}\prod_{i=1}^{n+1}(1 + 1/i) &= \left(1 + \frac{1}{n+1}\right)\prod_{i=1}^{n}(1 + 1/i) \\ &= \left(1 + \frac{1}{n+1}\right)(n + 1) \text{ (by the inductive hypothesis)} \\ &= n + 2\end{aligned}$$

as needed.

Induction, Formally

Induction

Let p be a predicate on integers. Suppose $p(n_0)$ for some integer n_0. Also suppose that for all $n \geq n_0$, $p(n)$ implies $p(n+1)$. It follows that $p(n)$ holds for all $n \geq n_0$.

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A proof by induction is usually broken into three parts

- Base case: Prove $p(n_0)$.
- Inductive hypothesis: Assume $p(n)$ for some arbitrary $n \geq n_0$.
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Note

Induction is one of the axioms included in the logical foundations of mathematics, though sometimes it is presented in an equivalent form called the “well-ordering” axiom. (See book)

Example: Odd Pie Fights

- An odd number of people $n \geq 3$ engage in a pie fight
- Each person has one pie
- Each person throws their pie at the closest other person
  - We assume there are no ties in distance.
- Show that there is one person who does not get hit.

Example: Odd Pie Fights

Proof

- Base case (3 people): The closest pair will exchange pies, and the third will not get hit.
- Inductive Hypothesis: When there are $n = 2k + 1$ people, someone doesn't get hit
- Inductive step: Assume the IH for $n = 2k + 1$, and prove that one person doesn't get hit in a pie fight with $n + 2 = 2(k + 1) + 1$. (Note, we are inducting on k)

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  - The closest pair a and b exchange pies.
  - Case 1: Of the remaining people, nobody throws a pie at a and b.
    - The remaining n people participate in a self-contained pie fight. By the inductive hypothesis, one of these n people does not get hit.

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  - The closest pair a and b exchange pies.
  - Case 1: Of the remaining people, nobody throws a pie at a and b.
    - The remaining n people participate in a self-contained pie fight. By the inductive hypothesis, one of these n people does not get hit.
  - Case 2: Some $c \neq a, b$ throws a pie at a or b.
    - One person gets hit by two pies. Since there are as many pies as people, someone must not get hit.

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Claim

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- Inductive Step:
  - Consider a $2^{k+1} \times 2^{k+1}$ chessboard with one square removed.
  - Split it into quadrants, each of which is $2^k \times 2^k$.
  - One of the quadrants is missing a square, and can be tiled by IH.
  - For the other three, remove three adjacent corner squares, tile by the IH, then add an L piece to fill in the corners.

Example: Pigeonhole Principle

See book

Outline

What is Strong Induction?

- Sometimes, to prove $p(n+1)$ you don't just need $p(n)$, but you need some previous values like $p(n-1)$, $p(n-3)$, etc.
- In our ladder analogy:
  - If you're carrying something heavy, you might need to push off both steps n and $n-1$ to get to step $n+1$.

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- In our domino analogy: Dominos might be heavy, need the last k dominos to exert enough force
- This is logically valid: So long as you've proven $p(n_0), \ldots p(n)$, you get to use any of them in your proof that $p(n+1)$.
- But you have to be careful to not accidentally use $p(m)$ for $m < n_0$
  - Sometimes you have to prove more base cases

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- Inductive step: To form $n + 1$, use IH to form $n - 3$ then add a 4 cent stamp.
  - Needed to "get the ball rolling" by proving a few base cases. Otherwise $n - 3$ "overshoots"

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- Induction Hypothesis: Any number between 2 and n can be written as a product of primes.
- Inductive step:
  - Consider $n + 1$. It is either prime or not.
  - If prime, we're done.
  - Otherwise, $n + 1 = ab$ for $2 \leq a, b \leq n$. By IH, each of a, b can be written as a product of primes. Therefore so can $n + 1$.

A Coin Game

See book

Strong Induction vs Induction

- It might appear that strong induction is stronger than induction
- However, this is an illusion: they are equivalent.
- Strong induction is just induction where the induction hypothesis involves universal quantification
- That said, sometimes it is easier to think in terms of strong induction.