CS170: Discrete Methods in Computer Science
Summer 2023
Propositional Logic

Instructor: Shaddin Dughmi[1]

[1]These slides adapt some content from similar slides by Aaron Cote. Moreover, the rules of inference table is drawn directly from those slides.

What is Logic?

- The language of mathematics!
- Symbols and rules for manipulating them
- Allows us to reason clearly:
  - Make precise statements
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- The language of mathematics!
- Symbols and rules for manipulating them
- Allows us to reason clearly:
  - Make precise statements
  - Derive new facts from old
- There are many sorts of logic, some more complicated and expressive than others
- Today: Propositional Logic (logic without quantification)
- Later in the class: First-order Logic (logic with quantification)

Outline

Proposition

A declarative statement of fact which is unambiguously either true or false.

Which of the following are propositions?

- 2000 was a leap year
- 2001 was a leap year
- 16 is a prime number
- 384921379417237 is a prime number
- Do your homework
- Colorless green ideas speak furiously
- This statement is false

Propositional Variables and Formulas

- We use variable symbols like p, q, r to refer to propositions
  - We call these propositional variables or atomic propositions.
- We can combine simple propositions to form more propositions using logical operators like $\neg$ (“not” a.k.a. “negation”), $\wedge$ (“and” a.k.a. “conjunction”), $\vee$ (“or” a.k.a. “disjunction”)
  - We call these propositional formulas or compound propositions
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Example

- $p =$ “there is life on earth”
- $q =$ “there is life on mars”
- $r =$ “there is life outside the solar system”
- $\neg p =$ “there is no life on Earth”
- $p \vee q =$ “there is life on Earth or on Mars (or both)”
- $p \wedge q =$ “there is life on Earth and on Mars”
- $(p \wedge q) \vee \neg r =$ “Either there is life on both Earth and Mars, or there is no life outside the solar system (or both). ”

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Note

We also allow the boolean constants T and F in formulas

- Convenient for proofs
- Not strictly necessary in a formula, since they can be simplified away

Truth Values and Truth Tables

- Each propositional variable can take value T (“True”) or F (“False”)
  - In digital logic, we sometimes use 1 for T and 0 for F
- A propositional formula’s truth value can be evaluated from the truth values of its atomic propositions
- This can be expressed as a truth table
  - A description of a boolean function which maps truth values of the variables to the truth value of the formula

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  - A description of a boolean function which maps truth values of the variables to the truth value of the formula
- Let's write the truth tables of the formulas from the last slide
- How many possible truth tables are there for n variables?

More Operators

- $\neg, \wedge, \vee$ are often thought of as the “basic” operators
  - They are really all you need to express any truth table
- However, some other operators are also common and useful
  - $p \oplus q$ (“Exclusive or”): Either p or q, but not both.
  - $p \Rightarrow q$ (“implies”): If p then q.
  - $p \iff q$ (“equivalence”): p if and only if q
  - $\ldots$

Let’s draw the truth tables defining these operators

More on the Implies Operator

- The implies operator is a very common and useful one
- It's worth reflecting on semantics of $p \Rightarrow q$:
  - $p \Rightarrow q$ can be written as $\neg p \vee q$
  - It's only false when p is true but q is false
  - True whenever p is false, regardless of what q is
  - A false assumption implies anything!

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  - It's only false when p is true but q is false
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  - A false assumption implies anything!
- $p \Rightarrow q$ is often read in variety of ways:
  - p implies q
  - If p then q
  - p only if q
  - q if p
  - q follows from p
  - q is necessary for p
  - p is sufficient for q
  - q unless $\neg p$

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You cannot ride the roller coaster if you are under 4 feet tall, unless you are older than 16 years.

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$$\neg o \Rightarrow (u \Rightarrow \neg r)$$

Propositional Formulas and Digital Logic

A hardware circuit simply evaluates a propositional formula!!

Outline

Properties of Individual Propositions

A (compound) proposition α is said to be

- satisfiable if there is a way to set its variables so it evaluates to true
  - At least one row of its truth table ends with a T
- unsatisfiable if it is not satisfiable.
  - All rows of its truth table ends with an F
  - e.g. $p \wedge \neg p$, $(\neg(p \Rightarrow q)) \wedge \neg p$
- a tautology if for any setting of its variables it evaluates to true
  - All rows of truth table end with a T
  - e.g. $p \vee \neg p$, $(p \Rightarrow q) \vee p$

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α is a tautology if and only if $\neg\alpha$ is unsatisfiable

Equivalence between Propositions

- Two propositions α and β are equivalent if they have the same truth value for every setting of the variables.
  - i.e., they have the same truth table
- We write $\alpha \equiv \beta$ to say that α and β are equivalent.
- E.g. $p \Rightarrow q \equiv \neg p \vee q$
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Note

$\equiv$ and $\iff$ are closely related, but are not the same! $\iff$ is part of the language of propositions, but $\equiv$ is a claim about propositions! In other words, $\alpha \iff \beta$ is a formula that may be true or false depending on how you set its variables, while $\equiv$ is a meta-statement asserting that two formulas have the same truth tables.

Consistency

A set of propositions is consistent if there is a way to set the variables so that all the propositions evaluate to true

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- There is life on Earth or on Mars
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Are the following consistent?

- There is life on Earth or on Mars
- If there is life on Earth, then there is life on Mars
- There is no life on Mars

Implication

Propositions $\alpha_1, \ldots, \alpha_k$ imply or entail proposition β if for every setting of the variables for which $\alpha_1, \ldots, \alpha_k$ evaluate to T, β evaluates to T as well.

- We also say β follows from $\alpha_1, \ldots, \alpha_k$.
- We call $\alpha_1, \ldots, \alpha_k$ the premises, and β the conclusion
- We write $\alpha_1, \ldots, \alpha_k \vdash \beta$
- E.g. $p \Rightarrow q, \quad p \vee q \quad \vdash \quad q$

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Note

The word “implies” is overloaded. One use of the word is for the logical operator $\Rightarrow$, and another is for $\vdash$. The former constructs a proposition that can be true or false, whereas the latter is a claim in a meta-language about one set of propositions entailing another.

Outline

Arguments

- An argument is a sequence of statements starting with premises (a.k.a. assumptions or axioms) and ending with a conclusion.
- When the argument is in in propositional logic, each statement is a propositional formula
- An argument is valid if each statement after the premises is logically implied by statements preceding it (in the sense of $\vdash$)

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Note

If the premises are inconsistent (i.e, inherently contradictory) then the argument is automatically valid! Once you prove F, you can prove anything! (Garbage in, garbage out)

Example: Valid Argument

- Premise: All men are mortal
- Premise: Socrates is a man
- Conclusion: Socrates is mortal

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Example: Invalid Argument

- Premise: If there is life on Earth then there is life on Mars
- Premise: Either there is life on Mars or there is life on Europa
- Premise: There is no life on Earth
- There is no life on Mars
- Conclusion: There is life on Europa

Proofs

A Proof is a valid argument where each statement after the premises "self-evidently" follows from the statements preceding it.

- For a formal proof, a self-evident step is one that uses one of the rules of inference of the logical system
- For a "proof", as the term is usually used, a self-evident step is one that your audience thinks is "obvious" or "easy".
- In a proof, your audience should have little trouble turning it into a formal proof with ample time and paper

Rules of Inference

- A rule of inference draws a logically valid conclusion from existing knowledge
- The rule is usually obvious or easy to check using a truth table
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Example: Hypothetical Syllogism

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore p \Rightarrow r \end{array}$$

Note: You apply this when p,q, r are formulas as well!

Rules of Inference

Rule	Meaning
Modus Ponens	p, p ⇒ q, then q
Modus Tollens	p ⇒ q, ¬q, then ¬p
Hypothetical Syllogism	p ⇒ q, q ⇒ r, then p ⇒ r
Disjunctive Syllogism	p ∨ q, ¬p, then q
Addition	p, then p ∨ q
Simplification	p ∧ q, then p
Conjunction	p, q, then p ∧ q
Resolution	p ∨ q, ¬p ∨ r, then q ∨ r

Rules of Inference (Equivalence)

Name	Meaning	Twin
Tautology	$p \lor \neg p \equiv T$	
Contradiction	$p \land \neg p \equiv F$	
Double Negation	$\neg(\neg p) \equiv p$	
Contrapositive	$p \Rightarrow q \equiv \neg q \Rightarrow \neg p$	
Mutual Implication	$p \Leftrightarrow q \equiv (p \Rightarrow q) \land (q \Rightarrow p)$	
Exclusive-or	$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$	
Implication	$p \Rightarrow q \equiv \neg p \lor q$	
Idempotent	$p \lor p \equiv p$	$p \land p \equiv p$
Identity	$F \lor p \equiv p$	$T \land p \equiv p$
Domination	$T \lor p \equiv T$	$F \land p \equiv F$
Commutative	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
Associative	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
Distributive	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
DeMorgan's	$\neg(p \land q) \equiv (\neg p \lor \neg q)$	$\neg(p \lor q) \equiv (\neg p \land \neg q)$
Absorption	$p \lor (p \land q) \equiv p$	$p \land (p \lor q) \equiv p$

Example

Starting from the premises $p \vee (q \wedge r)$ and $\neg r$, prove p.

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1. $p \vee (q \wedge r)$ (Premise)
2. $\neg r$ (Premise)
3. $(p \vee q) \wedge (p \vee r)$ (1, Distributive)
4. $(p \vee r) \wedge (p \vee q)$ (3, Commutative)
5. $p \vee r$ (4, Simplification)
6. $r \vee p$ (5, Commutative)
7. p (2 and 6, Disjunctive Syllogism)

Example

Show that $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$.

Example

Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.
We need to show that $\neg(p \vee (\neg p \wedge q))$ implies $\neg p \wedge \neg q$, and vice versa.

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Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.
We need to show that $\neg(p \vee (\neg p \wedge q))$ implies $\neg p \wedge \neg q$, and vice versa.
Let's start with the forward direction.

1. $\neg(p \vee (\neg p \wedge q))$ (Premise)
2. $\neg p \wedge \neg(\neg p \wedge q)$ (1,DeMorgan's)
3. $\neg p$ (2, Simplification)
4. $\neg(\neg p \wedge q) \wedge \neg p$ (2,Commutative)
5. $\neg(\neg p \wedge q)$ (4, Simplification)
6. $\neg\neg p \vee \neg q$ (5, DeMorgan's)
7. $\neg\neg\neg p$ (3, Double Negation)
8. $\neg q$ (6 and 7, Disjunctive Syllogism)
9. $\neg p \wedge \neg q$ (3 and 8, Conjunction)

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7. $\neg\neg\neg p$ (3, Double Negation)
8. $\neg q$ (6 and 7, Disjunctive Syllogism)
9. $\neg p \wedge \neg q$ (3 and 8, Conjunction)

For the backwards direction, we have to start with premise $\neg p \wedge \neg q$ and prove conclusion $\neg(p \vee (\neg p \wedge q))$. Left as an exercise.

Another Approach to Proving Equivalence

To show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$, we can manipulate using only logical equivalences. (No need for two directions anymore)

1. $\neg(p \vee (\neg p \wedge q))$
2. $\neg p \wedge \neg(\neg p \wedge q))$ (1,DeMorgan's)
3. $\neg p \wedge (p \vee \neg q)$ (2, DeMorgan's)
4. $(\neg p \wedge p) \vee (\neg p \wedge \neg q)$ (3, Distributive)
5. $F \vee (\neg p \wedge \neg q)$ (4, Contradiction)
6. $\neg p \wedge \neg q$ (5, Identity)

Rules of Inference vs Equivalences

- Some of the rules of inference can only be applied in one direction (Everything on our first list, e.g. Addition or Hypothetical Syllogism)
- Others go both ways (Everything on our second list, E.g. DeMorgan's), we call these Logical Equivalences.

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- Others go both ways (Everything on our second list, E.g. DeMorgan's), we call these Logical Equivalences.
- Equivalences can be used to manipulate a subformula of a statement you have in your proof (like in the previous slides)
  - E.g. $p \vee \neg(p \Rightarrow q) \equiv p \vee \neg(\neg q \Rightarrow \neg p)$ (Contrapositive)
- Rules of inference that are not equivalences cannot be used that way in general
  - E.g. $\neg p$ does not imply $\neg(p \vee q)$ by using the Addition rule

Garbage In, Garbage Out

Show that if you assume a statement p and its negation, then you can prove any other (possibly unrelated) statement q.

1. p (premise)
2. $\neg p$ (premise)
3. $p \wedge \neg p$ (1 and 2, Conjunction)
4. $(p \wedge \neg p) \vee q$ (3, Addition)
5. $F \vee q$ (4, Contradiction)
6. q (5, Identity)

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5. $F \vee q$ (4, Contradiction)
6. q (5, Identity)

More generally, if your premises are inconsistent then you can prove something and its negation, and therefore can prove anything.

Soundness and Completeness

There are two main desirable properties of a logical system

1. Soundness: You can only prove statements that are entailed by the assumptions
   - If you can write a formal proof that starts from premises A and ends with conclusion C, then every truth assignment that satisfies A must also satisfy C.
2. Completeness: Everything that is logically entailed by a set of assumptions can be formally proved
   - If it is indeed the case that every truth assignment that satisfies A also satisfies C, then there is a proof that starts with premises A and concludes C.

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2. Completeness: Everything that is logically entailed by a set of assumptions can be formally proved
   - If it is indeed the case that every truth assignment that satisfies A also satisfies C, then there is a proof that starts with premises A and concludes C.

   When designing a logic, it is trivial to have only one of these properties (why?). Takes more care to have both.

Luckily

Propositional Logic, with the rules of inference we saw, is both sound and complete!