Homework #1
CS599 Fall 2012

Due Friday 10/5

General Instructions The following problems are meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. Additionally, please provide mathematical proofs of all claims you make in your solutions.

Problem 1. Nash Equilibria in a Location Game.
Consider a walking trail along a stretch of beach, which we model as the interval $T = [0, 1]$. There are k ice-cream vendors, each looking to set up an icecream stand at some point along the trail, with the goal of attracting as many customers as possible. We assume that pedestrians out for a walk are uniformly distributed along the trail, and when faced with a craving for some icecream head to the nearest icecream stand, so that the utility of an icecream stand i is the measure of the set of points on the trail closer to i than to any of i's competitors. When two or more icecream stands are located at the same point p, they evenly split the customers closer to p than any of the other locations of icecream stands.

Formally, each icecream vendor $i \in \{1, \ldots, k\}$ chooses a location $x(i) \in [0, 1]$ — i.e. his space of actions is $[0, 1]$. Given $x \in [0, 1]^k$ and $i \in \{1, \ldots, k\}$, let $[l_x(i), r_x(i)]$ be the line segment of points in T no further from from $x(i)$ than from any other $x(j)$. Moreover, let $c_x(i)$ be the number of vendors co-located with i — i.e. the number of vendors j with $x(j) = x(i)$ (including i himself). The utility of player i is then defined as

$$u_i(x) = (r_x(i) - l_x(i))/c_x(i)$$

Clearly, this defines a game of complete information, where each player's strategy set is $[0, 1]$. Answer the following:

a. (3 points). What is the set of pure Nash equilibria when $k = 2$?

b. (3 points). What is the set of pure Nash equilibria for $k = 3$?

Now, consider a different trail T which goes around a circular lake — i.e. T is the circle in 2 dimensional space with circumference 1. The distance between two points p_1 and p_2 on T is naturally defined as the length of the shorter of the two arcs defined by p_1 and p_2. Define utilities as before: given a set of locations along T for each of the vendors, the utility of vendor i is the length of the arc of points on T which are no further from i than from any of i's competitors, divided by the number of vendors co-located with i. Answer the following:

c. (3 points). What is the set of pure Nash equilibria when $k = 2$?

d. (3 points). What is the set of pure Nash equilibria when $k \geq 3$?

Problem 2. Zero-sum Games and Linear Programming Review.

A game in normal form is said to be a zero-sum game if, for every action profile of the players, the sum of players' utilities is 0. Two-player zero-sum games exhibit specialized structure that renders them central to much of computer science and mathematical optimization. Specifically, zero-sum games are intimately related to *linear programming*, which is arguably *the* cornerstone of mathematical and combinatorial optimization. A canonical example of two-player zero-sum games is rock-paper-scissors.

Since the payoff of the second player is simply the negation of the payoff of the first player, we represent a two-player zero-sum game simply by a matrix $A \in \mathbb{R}^{n \times m}$, where A_{ij} is the payoff of player 1 when player 1 plays strategy i and player 2 plays strategy j. Let $\Delta_k = \left\{ p \in \mathbb{R}^k_+ : \sum_{i=1}^k p_i = 1 \right\}$ denote the simplex of dimension k, let A_j denote the j'th column of A, and let $\overline{A}_i$ denote the i'th row of A. Define the following two quantities:

$$maxmin(A) = \max_{p \in \Delta_n} \min_{j=1}^{m} p^T A_j$$

$$minmax(A) = \min_{q \in \Delta_m} \max_{i=1}^{n} \overline{A}_i q$$

Observe that $maxmin(A)$ is player 1's maximum, over all his mixed strategies p, of his utility when he plays p and player 2 plays a best response to p. The p achieving the maximum in this definition is referred to as player 1's maxmin strategy. Similarly, $minmax(A)$ is player 2's minimum, over all his mixed strategies q, of his loss (i.e. negation of his utility) when he plays q and player 1 best responds to q. The q achieving this minimum is referred to as player 2's maxmin strategy.

Groundwork. Review linear programming and linear programming duality. I recommend Luca Trevisan's notes here (chapters 5 and 6), though there are many good treatments of LP both in course notes and in textbooks. If you are confused, or want pointers to additional resources, come see me or write me.

a. Prove that, for every finite matrix A, $maxmin(A) = minmax(A)$. This is known as the *minimax theorem*, and the quantity $maxmin(A) = minmax(A)$ is referred to as *the value of the game.* You will need to invoke linear programming duality. **(Do not hand in)**

b. Prove that both players playing their maxmin strategies is a mixed Nash equilibirum of the zero-sum game. **(Do not hand in)**

c. (10 points). Consider the following 2-player, zero-sum game, defined on an undirected graph $G = (V, E)$ with a designated source node s, and target node t. The first player is the *attacker*, and his actions are the set of $s - t$ paths in G. Another player is the *defender*, and his actions are the set of edges in G. When the attacker plays a path p and the defender plays an edge e, the utility of the attacker is 1 if $e \notin p$ and 0 if $e \in p$. Find the value of the game. You may express the value of the game in terms of the size of the minimum s-t cut in G. (Hint: Invoke the minimax theorem and Menger's theorem (google it)).

Problem 3. Existence of Bayes-Nash Equilibrium. (12 points).
Prove that every finite Bayesian game of incomplete information, under the common prior assumption, admits a (mixed) Bayes-Nash equilibrium. (Hint: invoke Nash's theorem, which states that every finite game of *complete* information admits a mixed Nash equilibrium)

Problem 4. Auctions in More General Environments.

Recall that the Vickrey auction is a dominant-strategy truthful mechanism which sells a single item to the player who values it most. Now, we will extend the Vickrey auction to settings where the seller has multiple copies of the item for sale. As in single-item allocation, we assume players exhibit quasi-linear utility in all these examples — i.e. a player's utility for an allocation and payment is his value for the allocation less his payment.

a. (3 points). Consider an auctioneer with k identical items for sale, and n players interested in at most one item each. As in the single-item allocation problem, each player i's type is a real number v_i, encoding his value for receiving at least 1 item. Assume that a player exhibits *unit demand*, meaning that his value for more than one item is the same as his value for a single item. Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum over all players of their value for the items they receive. Your mechanism may charge payments, as in the Vickrey auction. (Hint: the mechanism will be a natural generalization of the Vickrey auction).

b. (Adapted from problem 2.3 in Hartline). (8 points). Now, consider a generalization of the k-item allocation problem in **(a)** above, faced in online advertising. Adwords is a Google product in which the search engine sells at auction advertisements that appear along side search results on the search results page.

Consider the following position auction environment which provides a simplified model of Adwords. There are m advertisement slots that appear along side search results and n advertisers (the players, or bidders). Each advertiser's type is his value v_i for a click on his advertisement. Slot j has *click through rate* w_j, meaning, if an advertiser is assigned slot j the advertiser will receive a click with probability w_j. Therefore, player i's value for being placed in slot j is $v_j w_j$. Each advertiser can be assigned at most one slot and each slot can be assignment at most one advertiser. You may assume, for convenience, that $w_1 \geq w_2 \geq \ldots \geq w_m$.

Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum over all players of their value per click multiplied by the click-through rate of the slot to which they are assigned, if any (else 0). As before, you may charge payments. (Hint: you can map the problem of welfare maximization in position auctions to m separate problems of welfare-maximization in k-item auctions (see subproblem **a**)).

c. (5 points). Now, consider different generalization of the k-item auction. A bipartite graph encodes sets of players who may jointly receive an item, as follows. Specifically, the bipartite graph G has players on the left hand side, and k non-identical items on the right hand side. The mechanism may assign item j to player i only if there is an edge between player i and item j in G. A player has the same value v_i, encoded by his type, for any item which may be assigned to him (as specified by G). Moreover, all players exhibit unit demand, as in subproblem **(b)**.

Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum, over all players i who receive an item, of the player's value v_i for an item. (Note: The k item auction of subproblem **(a)** is encoded by the complete bipartite graph.)