CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 16: Combinatorial Problems as Linear Programs II

Instructor: Shaddin Dughmi

Announcements

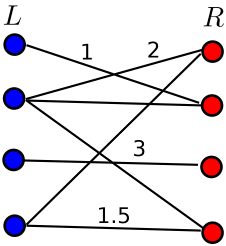
- Project announced
  - Choose a topic and partner(s) by Nov 1
  - Choose papers by Nov 8
  - Short report (≤ 10 pages) by Dec 6
- Today: Another case study on use LPs to encode combinatorial problems, featuring bipartite matching

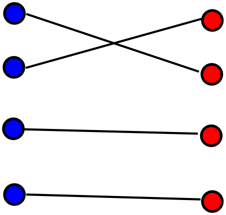
Outline

The Max-Weight Bipartite Matching Problem

Given a bipartite graph $G = (V, E)$, with $V = L \bigcup R$, and weights w_e on edges e, find a maximum weight matching.

- Matching: a set of edges covering each node at most once
- We use n and m to denote $|V|$ and $|E|$, respectively.
- Equivalent to maximum weight / minimum cost perfect matching.





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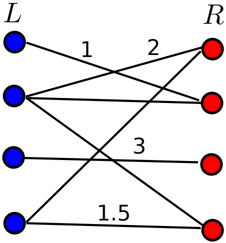

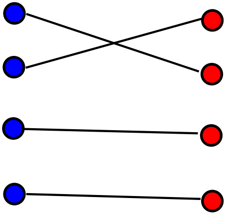

Our focus will be less on algorithms, and more on using polyhedral interpretation to gain insights about a combinatorial problem.

An LP Relaxation of Bipartite Matching

Bipartite Matching LP

max $\sum_{e \in E} w_e x_e$
s.t.

$$\sum_{e \in \delta(v)} x_e \leq 1, \qquad \forall v \in V.$$

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- Feasible region is a polytope $\mathcal{P}$ (i.e. a bounded polyhedron)
- This is a relaxation of the bipartite matching problem
  - Integer points in $\mathcal{P}$ are the indicator vectors of matchings.

$$\mathcal{P} \cap \mathbb{Z}^m = \{x_M : M \text{ is a matching}\}$$

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Integrality of the Bipartite Matching Polytope

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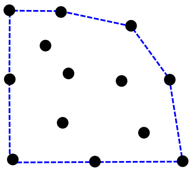

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The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

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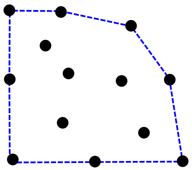

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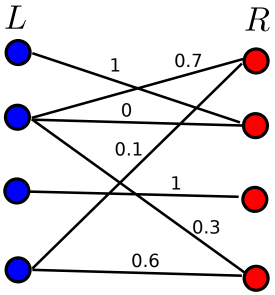
Note

- This is the strongest guarantee you could hope for of an LP relaxation of a combinatorial problem
- Solving LP is equivalent to solving the combinatorial problem
- Stronger guarantee than shortest path LP from last time

Proof



- Suffices to show that all vertices are integral (why?)

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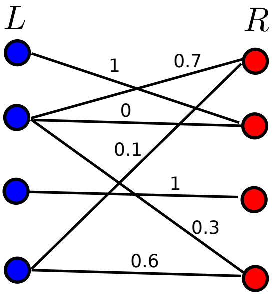

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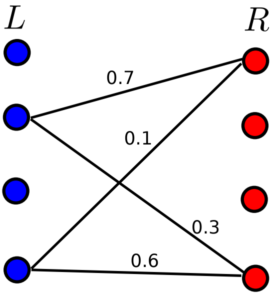

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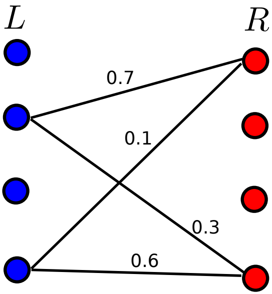

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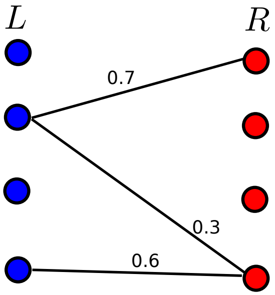

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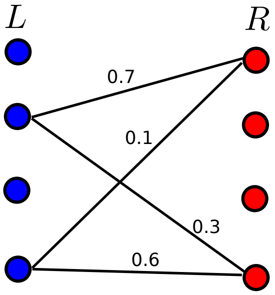
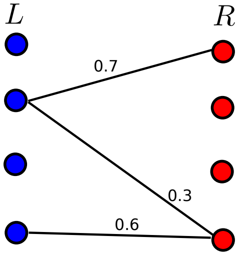


Case 1: Cycle C

- Let $C = (e_1, \ldots, e_k)$, with k even
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, +1, -1)$ to x_C preserves feasibility
- x is the midpoint of $x + \epsilon(+1, -1, ..., +1, -1)_C$ and $x - \epsilon(+1, -1, \ldots, +1, -1)_C$, so x is not a vertex.

Proof



Case 2: Maximal Path P

- Let $P = (e_1, \ldots, e_k)$, going through vertices $v_0, v_1, \ldots, v_k$
- By maximality, e_1 is the only edge of v_0 with non-zero x-weight
  - Similarly for e_k and v_k.
- There is $\epsilon > 0$ such that adding $\pm\epsilon(+1, -1, \ldots, ?1)$ to x_P preserves feasibility
- x is the midpoint of $x + \epsilon(+1, -1, ..., ?1)_P$ and $x - \epsilon(+1, -1, \ldots, ?1)_P$, so x is not a vertex.

Related Fact: Birkhoff Von-Neumann Theorem

$$\begin{aligned} \sum_{e \in \delta(v)} x_e = 1, &\quad \forall v \in V. \\ x_e \geq 0, &\quad \forall e \in E. \end{aligned}$$

- The analogous statement holds for the perfect matching LP above, by an essentially identical proof.

Related Fact: Birkhoff Von-Neumann Theorem

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Birkhoff Von-Neumann Theorem

The set of $n \times n$ doubly stochastic matrices is the convex hull of $n \times n$ permutation matrices.

e.g.

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} = 0.5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Outline

Total Unimodularity

We could have proved integrality of the bipartite matching LP using a more general tool

Definition

A matrix A is Totally Unimodular if every square submatrix has determinant 0, $+1$ or -1.

Theorem

If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, and b is an integer vector, then $\{x : Ax \leq b, x \geq 0\}$ has integer vertices.

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Proof

- Non-zero entries of vertex x are solution of $A'x' = b'$ for some nonsingular square submatrix A' and corresponding sub-vector b'
- Cramer's rule:

$$x'_i = \frac{\det(A'_i|b')}{\det A'}$$

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The constraint matrix of the bipartite matching LP is totally unimodular.

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- $A_{ve} = 1$ if e incident on v, and 0 otherwise.
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- If all columns of A' have two 1's,
  - Partition rows (vertices) into L and R
  - Sum of rows L is $(1, 1, \ldots, 1)$, similarly for R
  - A' is singular, so $\det A' = 0$.

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Primal and Dual LPs

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$$y_u + y_v \geq w_e, \quad \forall e = (u,v) \in E.$$

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- Primal interpertation: Player 1 looking to build a set of projects
  - Each edge e is a project generating "profit" w_e
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- Dual interpretation: Player 2 looking to buy resources
  - Offer a price y_v for each resource.
  - Prices should incentivize player 1 to sell resources
  - Want to pay as little as possible.

Vertex Cover Interpretation

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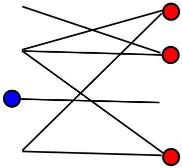
When edge weights are 1, binary solutions to dual are vertex covers

Definition

$C \subseteq V$ is a vertex cover if every $e \in E$ has at least one endpoint in C



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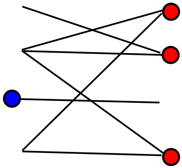

- Dual is a relaxation of the minimum vertex cover problem for bipartite graphs.
- By weak duality: min-vertex-cover $\geq$ max-cardinality-matching

König's Theorem

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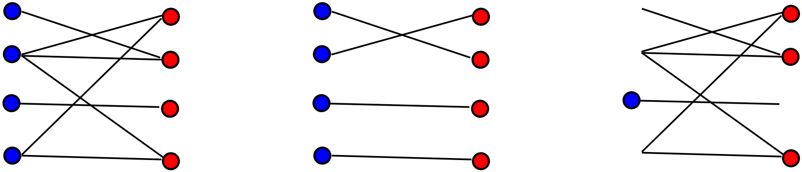
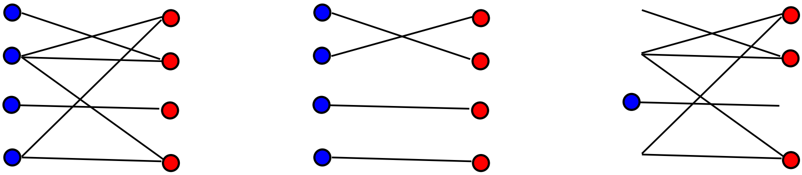
König's Theorem

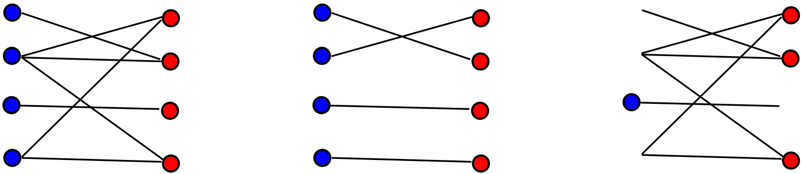
In a bipartite graph, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.

i.e. the dual LP has an optimal integral solution





- Let $M(G)$ be a max cardinality of a matching in G
- Let $C(G)$ be min cardinality of a vertex cover in G
- We already proved that $M(G) \leq C(G)$
- We will prove $C(G) \leq M(G)$ by induction on number of nodes in G.



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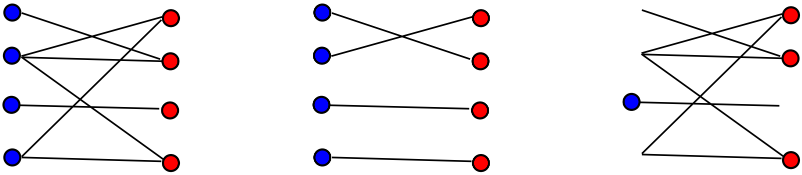

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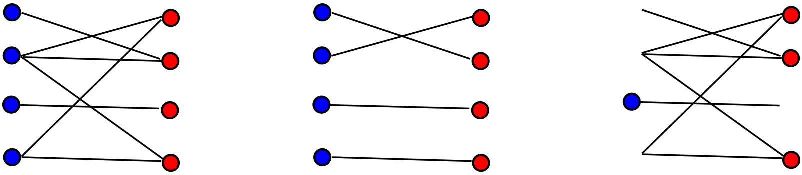

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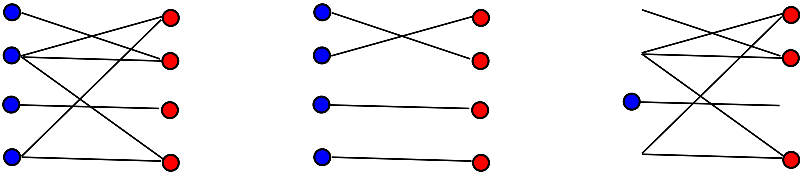

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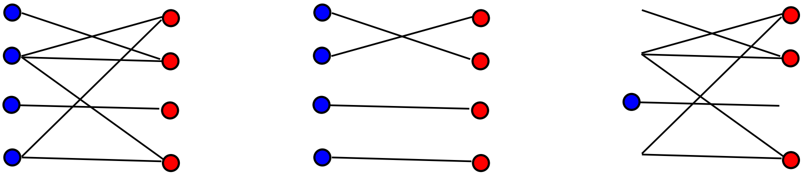

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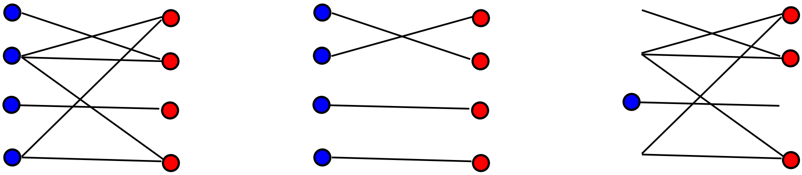

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Note: Could have proved the same using total unimodularity

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- Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time
- The same is true for the maximum independent set problem in bipartite graphs.
  - C is a vertex cover iff $V \setminus C$ is an independent set.