CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 2: Linear Programming Duality

Instructor: Shaddin Dughmi

Announcements

- Student Information
- Shaddin's Office Hours: Tuesdays 3:30pm - 4:30pm
- Books: Both available online. Will post links.
- This week's reading: Trevisan and Plotkin Lecture Notes

Outline

Linear Programming Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y = c \\ & y \geq 0 \end{aligned}$$

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- y_i is the dual variable corresponding to primal constraint $A_i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint corresponding to primal variable x_j

Linear Programming Duality: Standard Form, and Visualization

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\mathsf{T} x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & y^\mathsf{T} b \\ \text{subject to} \quad & A^\mathsf{T} y \geq c \\ & y \geq 0 \end{aligned}$$

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	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
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- y_i is the dual variable corresponding to primal constraint $A_i x \le b_i$
- $A_j^T y \ge c_j$ is the dual constraint corresponding to primal variable x_j

Interpretation 1: Economic Interpretation

Recall the Optimal Production problem from last lecture

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Facility wants to maximize profit subject to available raw materials

Interpretation 1: Economic Interpretation

Primal LP

$$
\begin{array}{lll}
\text{max} & \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\
& x_j \geq 0, & \text{for } j \in [n].
\end{array}
$$

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- Dual variable y_i is a proposed price per unit of raw material i
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials

Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

$$
\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 2 \\
& 2x_1 + x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.

Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

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- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.
- What if, instead of finding the optimal solution, we saught to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by $\frac{1}{3}$ and summing gives $x_1 + x_2 \leq 4/3$.

Interpretation 2: Finding the Best Upperbound

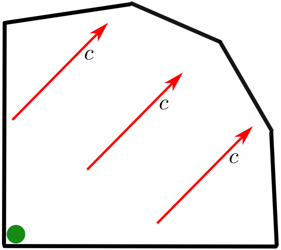
	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

- Multiplying each row i by y_i and summing gives the inequality

$$y^T A x \leq y^T b$$

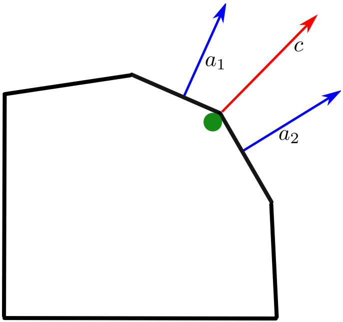
- When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$.
- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.

Interpretation 3: Physical Forces



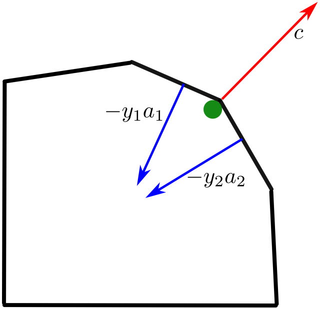
- Apply force field c to a ball inside bounded polytope $Ax \leq b$.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_ix \leq b_i$ applies some force $-y_ia_i$ to the ball

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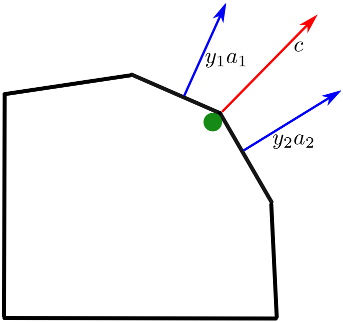
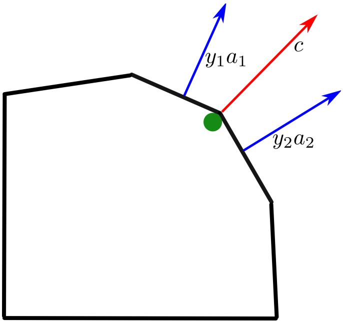


- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- Dual can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)

Outline

Duality is an Inversion

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.

Correspondance Between Variables and Constraints

Primal LP

$$
\begin{array}{lll}
\text{max} & \sum_{j=1}^{n} c_j x_j & \\
\text{s.t.} & & \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\
& x_j \geq 0, & \text{for } j \in [n].
\end{array}
$$

Dual LP

$$
\begin{array}{lll}
\text{min} & \sum_{i=1}^{m} b_i y_i & \\
\text{s.t.} & & \\
& \sum_{i=1}^{m} a_{ij} y_i \geq c_j, & \text{for } j \in [n]. \\
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Correspondance Between Variables and Constraints

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$$\begin{array}{llll} \text{max} & \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.} & & & \\ y_i: & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\ & x_j \geq 0, & \text{for } j \in [n]. \end{array}$$

Dual LP

$$\begin{array}{llll} \text{min} & \sum_{i=1}^{m} b_i y_i & & \\ \text{s.t.} & & & \\ & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, & \text{for } j \in [n]. \\ & y_i \geq 0, & \text{for } i \in [m]. \end{array}$$

- The i'th primal constraint gives rise to the i'th dual variable y_i

Correspondance Between Variables and Constraints

Primal LP

max	$\sum_{j=1}^{n} c_j x_j$	
s.t.		
y_i :	$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,$	for $i \in [m]$.
	$x_j \geq 0,$	for $j \in [n]$.

Dual LP

min	$\sum_{i=1}^{m} b_i y_i$	
s.t.		
x_j :	$\sum_{i=1}^{m} a_{ij} y_i \geq c_j,$	for $j \in [n]$.
	$y_i \geq 0,$	for $i \in [m]$.

- The i'th primal constraint gives rise to the i'th dual variable y_i
- The j'th primal variable x_j gives rise to the j'th dual constraint

Syntactic Rules

Primal LP

$$\begin{array}{lll} \max & c^\intercal x & \\ \text{s.t.} & & \\ y_i: & a_i x \leq b_i, & \text{for } i \in \mathcal{C}_1. \\ y_i: & a_i x = b_i, & \text{for } i \in \mathcal{C}_2. \\ & x_j \geq 0, & \text{for } j \in \mathcal{D}_1. \\ & x_j \in \mathbb{R}, & \text{for } j \in \mathcal{D}_2. \end{array}$$

Dual LP

$$\begin{array}{lll} \min & b^\intercal y & \\ \text{s.t.} & & \\ x_j: & \overline{a}_j^\intercal y \geq c_j, & \text{for } j \in \mathcal{D}_1. \\ x_j: & \overline{a}_j^\intercal y = c_j, & \text{for } j \in \mathcal{D}_2. \\ & y_i \geq 0, & \text{for } i \in \mathcal{C}_1. \\ & y_i \in \mathbb{R}, & \text{for } i \in \mathcal{C}_2. \end{array}$$

Rules of Thumb

- Loose constraint (i.e. inequality) $\Rightarrow$ tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) $\Rightarrow$ loose dual variable (i.e. unconstrained)

Outline

Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \leq b^\intercal y$.

Corollary

- *If primal and dual both feasible and bounded,* $OPT(Primal) \leq OPT(Dual)$
- *If primal is unbounded, dual is infeasible*
- *If dual is unbounded, primal is infeasible*

Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \ge c \\ & y \ge 0 \end{aligned}$$

Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \le b^\intercal y$.

Corollary

If x is primal feasible, and y is dual feasible, and $c^\intercal x = b^\intercal y$, then both are optimal.

Interpretation of Weak Duality

Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

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Upperbound Interpretation

Self explanatory

Interpretation of Weak Duality

Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

Upperbound Interpretation

Self explanatory

Physical Interpretation

Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.

Proof of Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

$$c^\intercal x \leq y^\intercal A x \leq y^\intercal b$$

Strong Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \geq c \\ & y \geq 0 \end{aligned}$$

Theorem (Strong Duality)

If either the primal or dual is feasible and bounded, then so is the other and $OPT(Primal) = OPT(Dual)$.

Interpretation of Strong Duality

Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

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Upperbound Interpretation

The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.

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Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

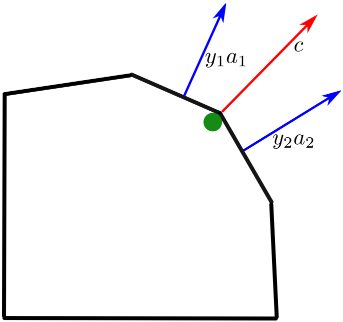
Upperbound Interpretation

The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.

Physical Interpretation

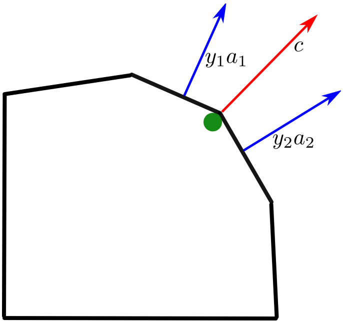
There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.

Informal Proof of Strong Duality



- Recall the physical interpretation of duality

Informal Proof of Strong Duality



- Recall the physical interpretation of duality
- When ball is stationary at x, we expect force c to be neutralized only by constraints that are tight
  - i.e. force multipliers y such that $y_i(b_i - a_i x) = 0$

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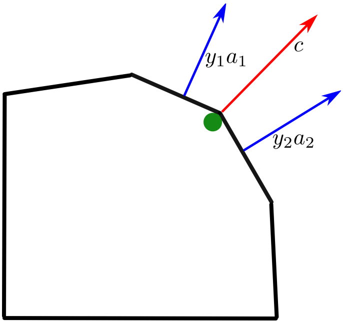

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  - i.e. force multipliers y such that $y_i(b_i - a_i x) = 0$

$$y^\intercal b - c^\intercal x = y^\intercal b - y^T A x = \sum_i y_i(b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!

Next Lecture

- Formal proof of Strong Duality
- Complementary slackness
- Sensitivity analysis
- Examples and applications of LP Duality