CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 20: Consequences of the Ellipsoid Algorithm

Instructor: Shaddin Dughmi

Announcements

Outline

1. Recapping the Ellipsoid Method
2. Complexity of Convex Optimization
3. Complexity of Linear Programming
4. Equivalence of Separation and Optimization

Recall: Feasibility Problem

The ellipsoid method solves the following problem.

Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing K
- A rational number $R > 0$ satisfying $\mathbf{vol}(E) \leq R$.
- A rational number $r > 0$ such that if K is nonempty, then $\mathbf{vol}(K) \geq r$.

Find a point $x \in K$ or declare that K is empty.

- Equivalent variant: drop the requirement on volume $\mathbf{vol}(K)$, and either find a point $x \in K$ or an ellipsoid $E \supseteq K$ with $\mathbf{vol}(E) < r$.

All the ellipsoid method needed was the following subroutine

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting x from K.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\intercal y \leq h^\intercal x\}$$

with x at its boundary.

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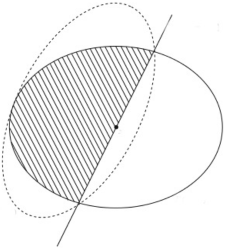
- The positive semi-definite cone S_n^+: Let H be the outer product $vv^\intercal$ of an eigenvector v of X corresponding to a negative eigenvalue.



Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output c.
   - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
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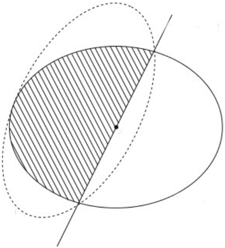

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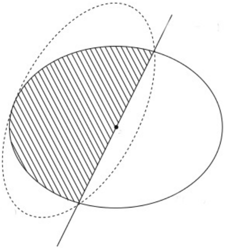

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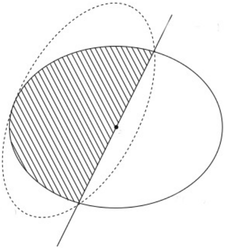

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Properties

Using T to denote the runtime of the separation oracle

Theorem

The ellipsoid algorithm terminates in time polynomial n, $\ln \frac{R}{r}$, and T, and either outputes $x \in K$ or correctly declares that K is empty.

We proved most of this. For the rest, see references.

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Note

For runtime polynomial in input size we need

- T polynomial in input size
- $\frac{R}{r}$ exponential in input size

Outline

Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex

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- Recall: A problem Π is a family of instances $I = (f, \mathcal{X})$
- When represented explicitly, often given in standard form

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \text{for } i \in \mathcal{C}_1. \\ & a_i^\intercal x = b_i, & \text{for } i \in \mathcal{C}_2. \end{array}$$

- The functions f,$\{g_i\}_i$ are given in some parametric form allowing evaluation of each function and its derivatives.

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- We will abstract away details of how instances of a problem are represented, but denote the length of the description by $\langle I \rangle$
- We simply require polynomial time (in $\langle I \rangle$ and n) separation oracles and such.

Solvability of Convex Optimization

There are many subtly different "solvability statements". This one is the most useful, yet simple to describe, IMO.

Requirements

We say an algorithm weakly solves a convex optimization problem in polynomial time if it:

- Takes an approximation parameter $\epsilon > 0$
- Terminates in time $\mathrm{poly}(\langle I \rangle, n, \log(\frac{1}{\epsilon}))$
- Returns an ϵ-optimal $x \in \mathcal{X}$:

$$f(x) \leq \min_{y \in \mathcal{X}} f(y) + \epsilon[\max_{y \in \mathcal{X}} f(y) - \min_{y \in \mathcal{X}} f(y)]$$

Solvability of Convex Optimization

Theorem (Polynomial Solvability of CP)

Consider a family Π of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

- *A separation oracle for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$*
- *A first order oracle for f: evaluates $f(x)$ and $\bigtriangledown f(x)$.*
- *An algorithm which computes a starting ellipsoid $E \supseteq \mathcal{X}$ with $\frac{\mathbf{vol}(E)}{\mathbf{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.*

Then there is a polynomial time algorithm which weakly solves Π.

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Let's now prove this, by reducing to the ellipsoid method

Proof (Simplified)

Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

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We can feed this into the Ellipsoid method!

Needed Ingredients

1. Separation oracle for new feasible set K:
2. Ellipsoid E containing K:
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

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$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

Lemma

$\mathbf{vol}(K) \leq \epsilon^n \, \mathbf{vol}(X)$.

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- Let $y = \epsilon x$ for $x \in \mathcal{X}$, and invoke Jensen's inequality

$$f(y) = f(\epsilon x + (1-\epsilon)0) \leq \epsilon f(x) + (1-\epsilon) f(0) \leq \epsilon$$

Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
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- We don't need to know anything about T!

Key Observation

We don't really need to know T, H, or L to simulate the same execution of the ellipsoid method on K!!

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- Require separation oracle for K to use $\bigtriangledown f$ only as a last resort
  - This is allowed.
  - Tries to get feasibility whenever possible.

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- Run this simulation until enough iterations have passed, and take the best feasible point encountered. This must be in K.

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A problem of maximizing a linear function over a polyhedron.

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  - Explicit: given by A, b and c
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- In the implicit case, we require polynomial time in the bit complexity of individual entries of A, b, c.

Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \leq b$: trivial when explicitly represented
- A first order oracle for $c^\intercal x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
- A way of "rounding" an ϵ-optimal solution to an optimal vertex solution: tricky

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Solution to both issues involves tedious accounting of numerical issues

Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

Lemma

Let v be vertex of the polyhedron $Ax \leq b$. It is the case that v has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

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Key to tackling both difficulties is the following observation:

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- Bounding ellipsoid: all vertices contained in the box $-2^M \leq x \leq 2^M$, which in turn is contained in an ellipsoid of volume exponential in M and n.

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- Volume lowerbound: Relaxing to $Ax \leq b + \epsilon$, for sufficiently small ϵ with $\langle \epsilon \rangle = \text{poly}(M)$. Gives volume exponentially small in M, but no smaller. Still close enough to original polyhedron so solution to relaxed problem can be “rounded” to solution of the latter.

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Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Rounding to a vertex: If a point y is ϵ-optimal, for sufficiently small ϵ chosen carefully to polynomial in description of input, then rounding to the nearest x with M bits recovers the vertex.

Theorem (Polynomial Solvability of Implicit LP)

Consider a family Π of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

- *A separation oracle for the polyhedron $Ax \leq b$*
- *Explicit access to c*

Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\text{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for Π (both primal and dual).

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  - It turns out this is still OK, but takes a lot of work.

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For the dual, we need equivalence of separation and optimization (HW?)

Outline

Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle
- As it turns out, the two tasks are polynomial-time equivalent.

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- As it turns out, the two tasks are polynomial-time equivalent.

Lets formalize the two questions, parametrized by a polytope P.

Linear Optimization Problem

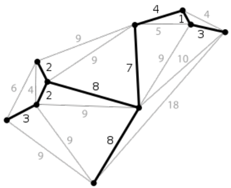
- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $\operatorname{argmax}_{x \in P} c^\intercal x$.

Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P$, or else find $h \in \mathbb{R}^n$ s.t. $h^\intercal x < h^\intercal y$ for all $x \in P$.

Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs c_e on edges e, find a minimum cost spanning tree of G.



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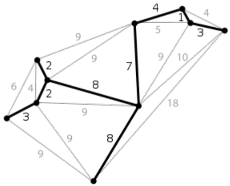

Spanning Tree Polytope

$$
\begin{aligned}
&\sum_{e \subseteq X} x_e \leq |X| - 1, && \text{for } X \subset V. \\
&\sum_{e \in E} x_e = n - 1 \\
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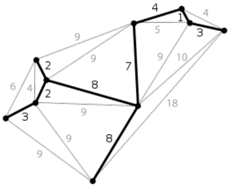

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- Optimization: Find the minimum/maximum weight spanning tree
- Separation: Find $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - i.e. When edge weights are x, find a “dense” subgraph

Theorem (Equivalence of Separation and Optimization for Polytopes)

Consider a family $\mathcal{P}$ *of polytopes* $P = \{x : Ax \leq b\}$ *described implicitly using* $\langle P \rangle$ *bits, and satisfying* $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$*. Then the separation problem is solvable in* $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ *time for* $P \in \mathcal{P}$ *if and only if the linear optimization problem is solvable in* $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ *time.*

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- E.g. Spanning tree polytopes, represented by graphs, are solvable.
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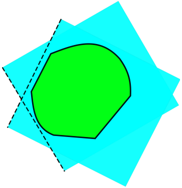
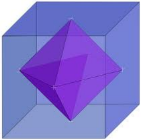
  - Separation $\Rightarrow$ optimization
- For the other direction, we need polars

Recall: Polar Duality of Convex Sets





One way of representing the all halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The polar of S is defined as follows:

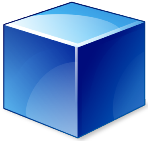
$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$

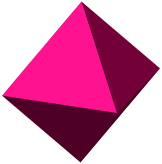
Note

- Every halfspace $a^\intercal x \leq b$ with $b \neq 0$ can be written as a "normalized" inequality $y^\intercal x \leq 1$, by dividing by b.
- S° can be thought of as the normalized representations of halfspaces containing S.

Properties of the Polar

1. If S is bounded and $0 \in interior(S)$, then the same holds for S°.
2. $S^{\circ\circ} = S$





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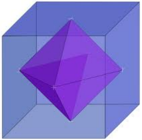
Polarity of Polytopes



Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A.

- Facets of P correspond to vertices of P°.
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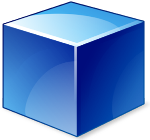
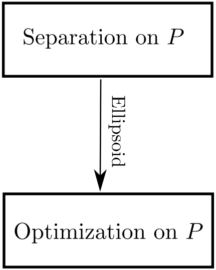
Proof Outline: Optimization ⇒ Separation





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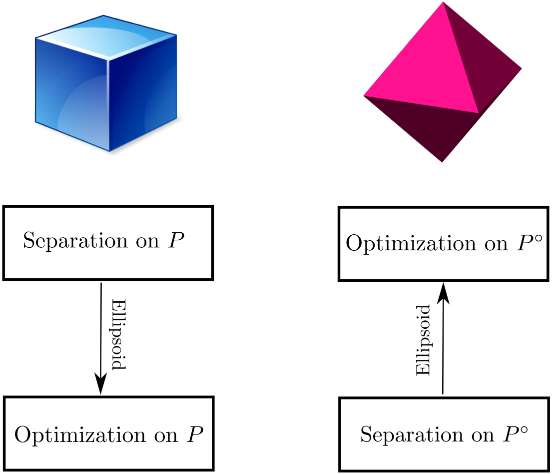

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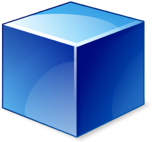

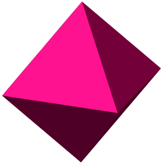

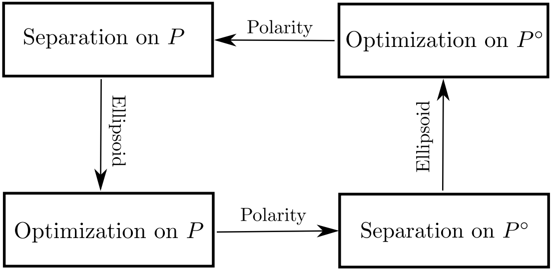

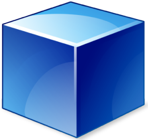

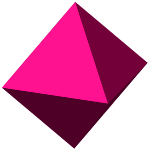

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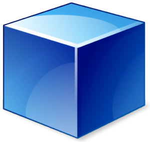

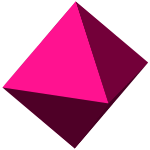

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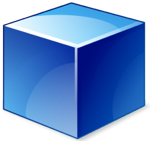
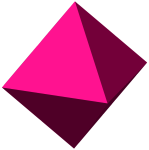
Separation over S reduces in constant time to optimization over S°, and vice versa since $S^{\circ\circ} = S$.





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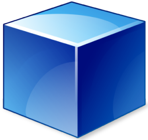

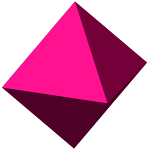

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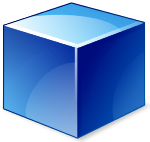

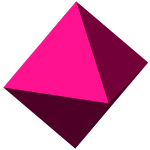

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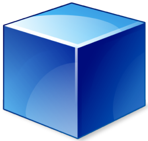

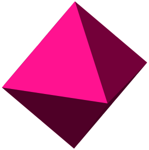

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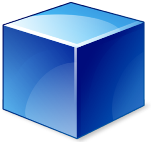
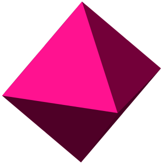
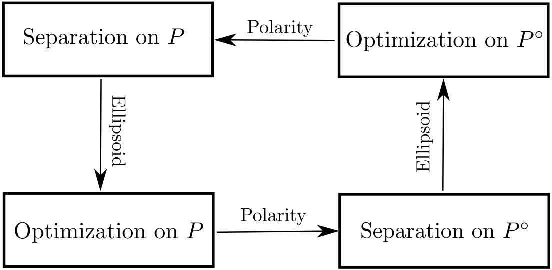
- If we find $y \in S^\circ$ s.t. $y \cdot x > 1$, then y is the separating hyperplane
  - $y^\intercal z \leq 1 < y^\intercal x$ for every $z \in S$.

Optimization $\Longleftrightarrow$ Separation







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Problems parametrized by P, a closed convex set.

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- Output: $x \in P^{+\epsilon}$, and $c^\intercal x \geq \max_{x' \in P} c^\intercal x' - \epsilon$

Weak Separation Problem

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- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P^{-\epsilon}$, or else find $h \in \mathbb{R}^n$ with $||h|| = 1$ s.t. $h^\intercal x < h^\intercal y + \epsilon$ for all $x \in P$.

I could have equivalently stated the weak optimization problem for convex functions instead of linear.

Theorem (Equivalence of Separation and Optimization for Convex Sets)

Consider a family $\mathcal{P}$ *of convex sets described implicitly using* $\langle P \rangle$ *bits. Then the weak separation problem is solvable in* $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ *time for* $P \in \mathcal{P}$ *if and only if the weak optimization problem is also solvable in* $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ *time.*

- The “approximation” in this statement is necessary, since we can’t solve convex optimization problems exactly.
- Weak separation suffices for ellipsoid, which is only approximately optimal anyways
- By polarity, weak optimization is equivalent to weak separation

Implication: Constructive Caratheodory

Implication: Solvability is closed under intersection