CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 24: Introduction to Submodular Functions

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Announcements

Introduction

- We saw how matroids form a class of feasible sets over which optimization of modular objectives is tractable
- If matroids are discrete analogues of convex sets, then submodular functions are discrete analogues of convex/concave functions
  - Submodular functions behave like convex functions sometimes (minimization) and concave other times (maximization)
- Today we will introduce submodular functions, go through some examples, and mention some of their properties

Outline

Set Functions

- A set function takes as input a set, and outputs a real number
  - Inputs are subsets of some ground set X
  - $f : 2^X \to \mathbb{R}$
- We will focus on set functions where X is finite, and denote $n = |X|$

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- We will focus on set functions where X is finite, and denote $n = |X|$
- Equivalently: map points in the hypercube $\{0, 1\}^n$ to the real numbers
  - Can be plotted as 2^n points in $n + 1$ dimensional space

Set Functions

- We have already seen modular set functions
  - Associate a weight w_i with each $i \in X$, and set $f(S) = \sum_{i \in S} w_i$
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- Supmodular/supermodular functions are weak analogues to convex/concave functions (in no particular order!)
- Other possibly useful properties a set function may have:
  - Monotone increasing or decreasing
  - Nonnegative: $f(A) \geq 0$ for all $S \subseteq X$
  - Normalized: $f(\emptyset) = 0$.

Submodular Functions

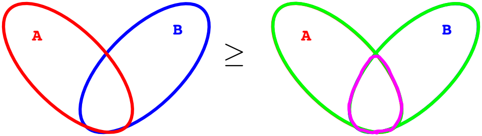
Definition 1

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

for all $A, B \subseteq X$.

- "Uncrossing" two sets reduces their total function value



Submodular Functions

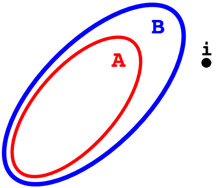
Definition 2

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$

for all $A \subseteq B \subseteq X$.

- The marginal value of an additional element exhibits "diminishing marginal returns"
- Should remind of concavity



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Examples

Many common examples are monotone, normalized, and submodular. We mention some.

Coverage Functions

X is the left hand side of a graph, and $f(S)$ is the total number of neighbors of S.

- Can think of $i \in X$ as a set, and $f(S)$ as the total “coverage” of S.

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Probability

X is a set of probability events, and $f(S)$ is the probability at least one of them occurs.

Examples

Social Influence

- X is the family of nodes in a social network
- A meme, idea, or product is adopted at a set of nodes S
- The idea propagates through the network through some random diffusion process
  - Many different models
- $f(S)$ is the expected number of nodes in the network which end up adopting the idea.

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Utility Functions

When X is a set of goods, $f(S)$ can represent the utility of an agent for a bundle of these goods. Utilities which exhibit diminishing marginal returns are natural in many settings.

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Clustering Quality

X is the set of nodes in a graph G, and $f(S) = E(S)$ is the internal connectedness of cluster S.

- Supermodular

Examples

There are fewer examples of non-monotone submodular/supermodular functions, which are nontheless fundamental.

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X is the set of nodes in a graph G, and $f(S)$ is the number of edges crossing the cut $(S, X \setminus S)$.

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- Neither submodular nor supermodular
- However, maximizing it reduces to maximizing supermodular function $E(S) - \alpha|S|$ for various $\alpha > 0$ (binary search)

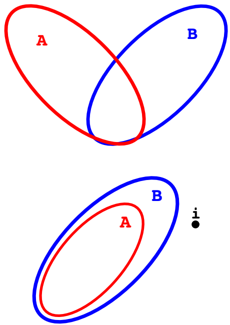
Equivalence of Both Definitions

Definition 1

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

Definition 2

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$



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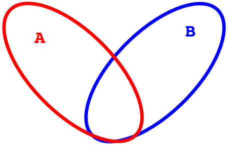

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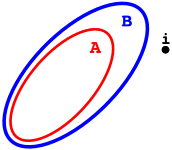

Definition 1 $\Rightarrow$ Definition 2

- To prove (2), let $A' = A \bigcup \{i\}$ and $B' = B$ and apply (1)

$$\begin{aligned} f(A \cup \{i\}) + f(B) &= f(A') + f(B') \\ &\geq f(A' \cap B') + f(A' \cup B') \\ &= f(A) + f(B \cup \{i\}) \end{aligned}$$

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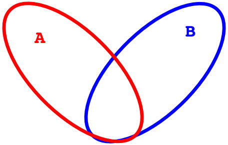

Definition 2

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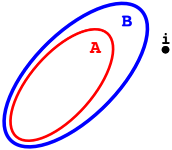

Definition 2 $\Rightarrow$ Definition 1

- To prove (1), start with $A = B$ and repeatedly elements to one but not the other
- At each step, (2) implies that the LHS of inequality (1) increases more than the RHS

Operations Preserving Submodularity

- Nonnegative-weighted combinations (a.k.a. conic combinations): If $f_1, \ldots, f_k$ are submodular, and $w_1, \ldots, w_k \geq 0$, then $g(S) = \sum_i w_i f_i(S)$ is also submodular
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Note

The minimum or maximum of two submodular functions is not necessarily submodular

Optimizing Submodular Functions

- As our examples suggest, optimization problems involving submodular functions are very common
- These can be classified on two axes: constrained/unconstrained and maximization/minimization

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) $O(1)$ ("nice" constriants)	Usually NP-hard to apx. Few easy special cases

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Representation

In order to generalize all our examples, algorithmic results are often posed in the value oracle model. Namely, we only assume we have access to a subroutine evaluating $f(S)$.