CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 7: Duality of Convex Sets and Functions

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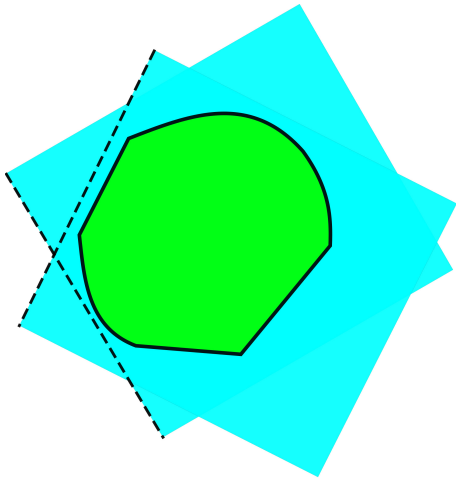
Announcements

- Today: Duality of Convex Sets and Functions
  - Read all of B&V Chapter 3.
  - Some material not in book. Will post references.

Outline

1. Convexity and Duality
2. Duality of Convex Sets
3. Duality of Convex Functions

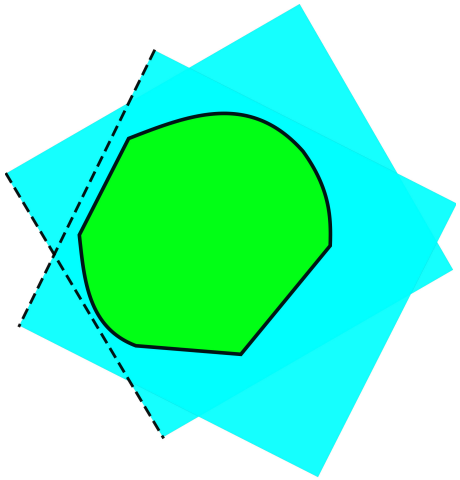
Duality Correspondances



There are two equivalent ways to represent a convex set

- The family of points in the set (standard representation)
- The set of halfspaces containing the set (“dual” representation)

Duality Correspondances



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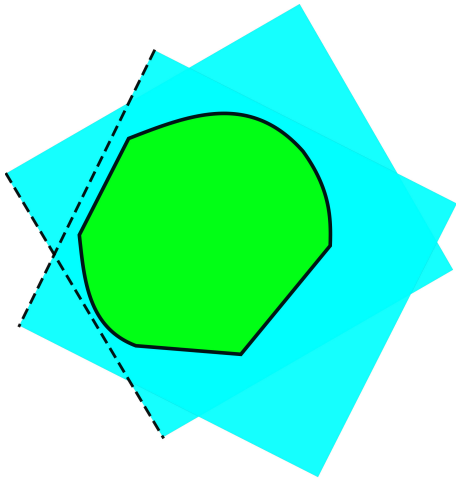
This equivalence between the two representations gives rise to a variety of “duality” relationships among convex sets, cones, and functions.

Duality Correspondances



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- The family of points in the set (standard representation)
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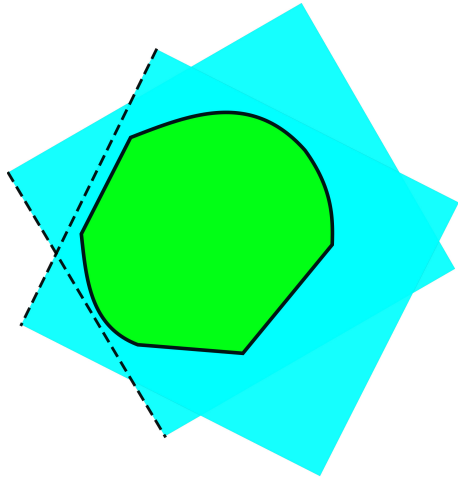
This equivalence between the two representations gives rise to a variety of "duality" relationships among convex sets, cones, and functions.

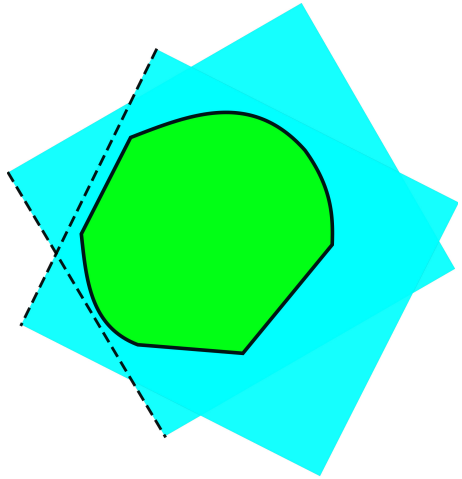
Definition

"Duality" is a woefully overloaded mathematical term for a relation that groups elements of a set into "dual" pairs.



Theorem

A closed convex set S is the intersection of all closed halfspaces $\mathcal{H}$ containing it.

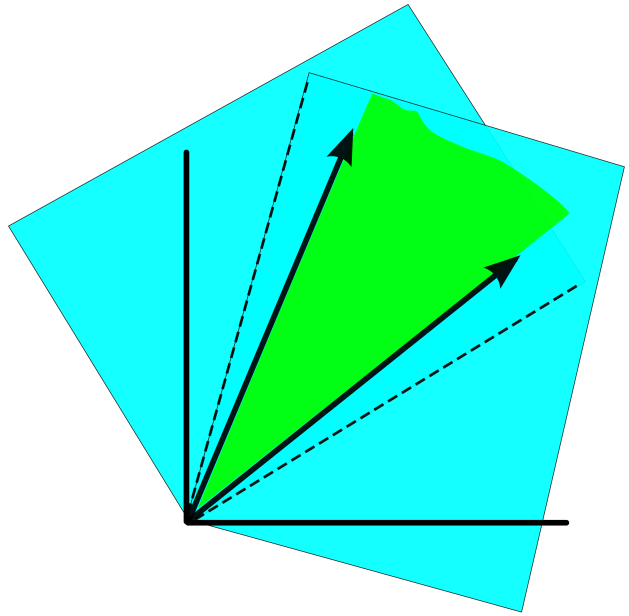


Theorem

A closed convex set S is the intersection of all closed halfspaces $\mathcal{H}$ containing it.

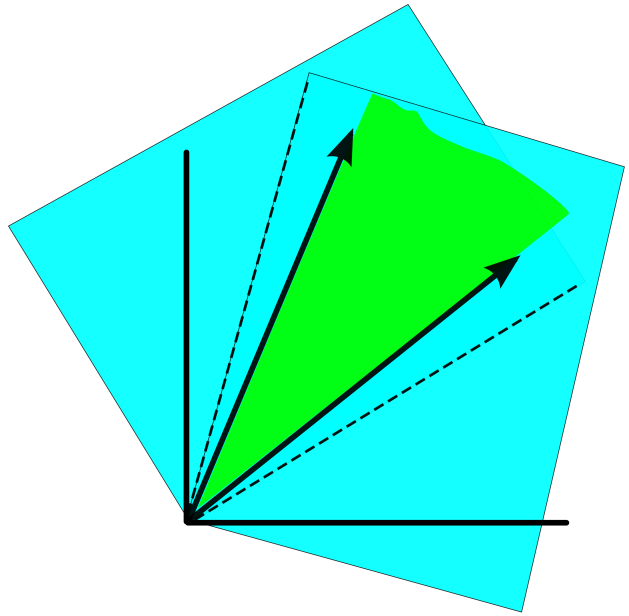
Proof

- Clearly, $S \subseteq \bigcap_{H \in \mathcal{H}} H$
- To prove equality, consider $x \notin S$
- By the separating hyperplane theorem, there is a hyperplane separating S from x
- Therefore there is $H \in \mathcal{H}$ with $x \notin H$, hence $x \notin \bigcap_{H \in \mathcal{H}} H$



Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.

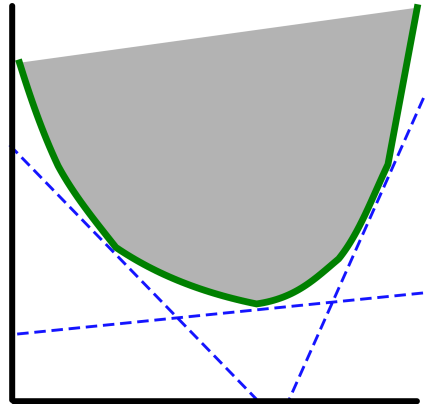


Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.

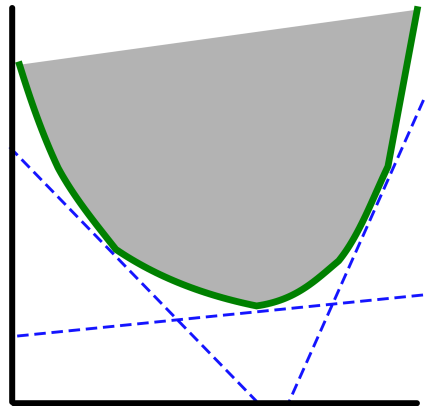
Proof

- For every non-homogeneous halfspace $a^\intercal x \leq b$ containing K, the smaller homogeneous halfspace $a^\intercal x \leq 0$ contains K as well.
- Therefore, can discard non-homogeneous halfspaces when taking the intersection



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.

Proof

- $\mathbf{epi}\, f$ is convex
- Therefore $\mathbf{epi}\, f$ is the intersection of a family of halfspaces of the form $a^\intercal x - t \leq b$, for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- Each such halfspace constrains $(x, t) \in \mathbf{epi}\, f$ to $a^\intercal x - b \leq t$
- $f(x)$ is the lowest t s.t. $(x, t) \in \mathbf{epi}\, f$
- Therefore, $f(x)$ is the point-wise maximum of $a^\intercal x - b$ over all halfspaces

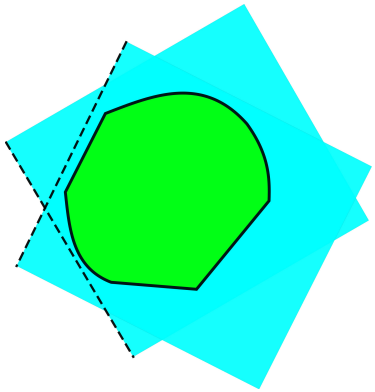
Outline

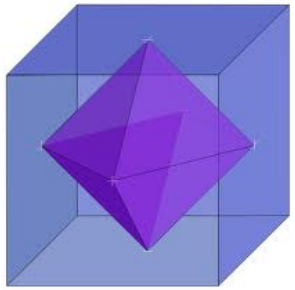
1 Convexity and Duality

2 Duality of Convex Sets

3 Duality of Convex Functions

Polar Duality of Convex Sets





One way of representing the all halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The polar of S is defined as follows:

$$S^\circ = \{y : y^\intercal x \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $a^\intercal x \leq b$ with $b \neq 0$ can be written as a "normalized" inequality $y^\intercal x \leq 1$, by dividing by b.
- S° can be thought of as the normalized representations of halfspaces containing S.

$$S^\circ = \{y : y^\intercal x \leq 1 \text{ for all } x \in S\}$$

Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

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Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

2. Follows from representation as intersection of halfspaces
3. S contains an ϵ-ball centered at the origin, so $||y|| \leq 1/\epsilon$ for all $y \in S^\circ$.

$$S^\circ = \{y : y^\intercal x \leq 1 \text{ for all } x \in S\}$$

Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

1.
   - Easy to see that $S \subseteq S^{\circ\circ}$
   - Take $x_\circ \notin S$, by SHT and $0 \in S$, there is a halfspace $z^\intercal x \leq 1$ containing S but not $x_\circ$ (i.e. $z^\intercal x_\circ > 1$)
   - $z \in S^\circ$, therefore $x \notin S^{\circ\circ}$

$$S^\circ = \{y : y^\intercal x \leq 1 \text{ for all } x \in S\}$$

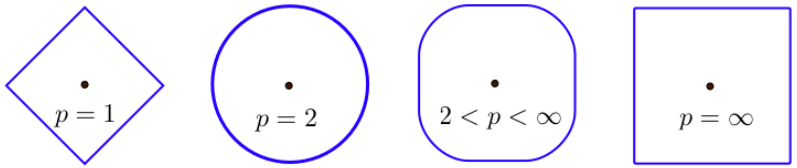
Properties of the Polar

1. $S^{\circ\circ} = S$
2. S° is a closed convex set containing the origin
3. When 0 is in the interior of S, then S° is bounded.

Note

When S is non-convex, $S^\circ = (convexhull(S))^\circ$, and $S^{\circ\circ} = convexhull(S)$.

Examples

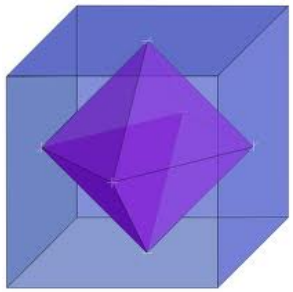


The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in $\mathbb{R}^2$.

Norm Balls

- The polar of the Euclidean unit ball is itself (we say it is self-dual)
- The polar of the 1-norm ball is the ∞-norm ball
- More generally, the p-norm ball is dual to the q-norm ball, where $\frac{1}{p} + \frac{1}{q} = 1$

Examples

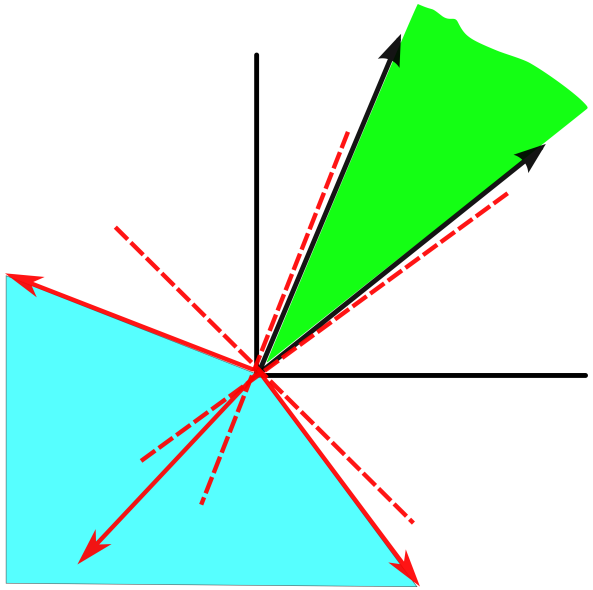


Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A.

- Facets of P correspond to vertices of P°.
- Dually, vertices of P correspond to facets of P°.

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

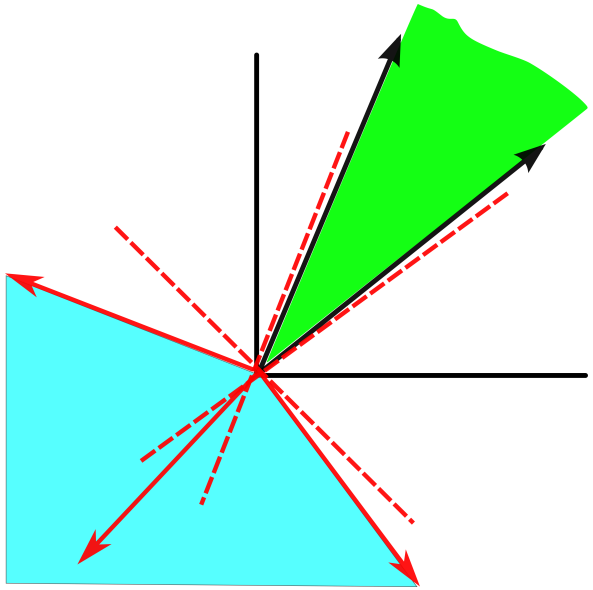
The polar of a closed convex cone K is given by

$$K^\circ = \{y : y^\intercal x \leq 0 \text{ for all } x \in K\}$$

Note

- If halfspace $y^\intercal x \leq b$ contains K, then so does smaller $y^\intercal x \leq 0$.
- K° represents all homogeneous halfspaces containing K.

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

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The polar of a closed convex cone K is given by

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Dual Cone

By convention, $K^* = -K^\circ$ is referred to as the dual cone of K.

$$K^* = \{y : y^\intercal x \geq 0 \text{ for all } x \in K\}$$

$$K^\circ = \{y : y^\mathsf{T} x \leq 0 \text{ for all } x \in K\}$$

Properties of the Polar Cone

1. $K^{\circ\circ} = K$
2. K° is a closed convex cone

$$K^\circ = \{y : y^\mathsf{T} x \leq 0 \text{ for all } x \in K\}$$

Properties of the Polar Cone

1. $K^{\circ\circ} = K$
2. K° is a closed convex cone

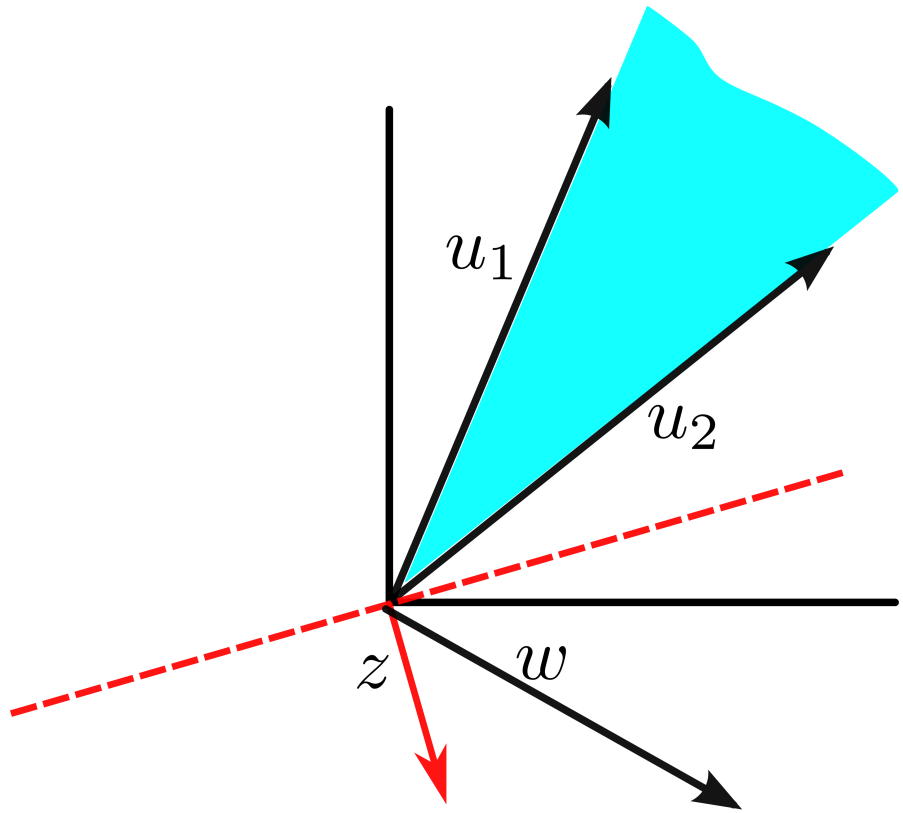
1. Same as before
2. Intersection of homogeneous halfspaces

Examples

- The polar of a subspace is its orthogonal complement
- The polar cone of the nonnegative orthant is the nonpositive orthant
  - Self-dual
- The polar of a polyhedral cone $Ax \leq 0$ is the conic hull of the rows of A
- The polar of the cone of positive semi-definite matrices is the cone of negative semi-definite matrices
  - Self-dual

Recall: Farkas' Lemma

Let K be a closed convex cone and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^\intercal x \leq 0$ for all $x \in K$, and $z^\intercal w > 0$.



Equivalently: there is $z \in K^\circ$ with $z^\intercal w > 0$.

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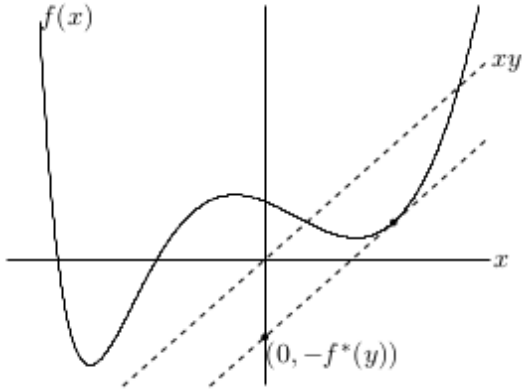
Conjugation of Convex Functions



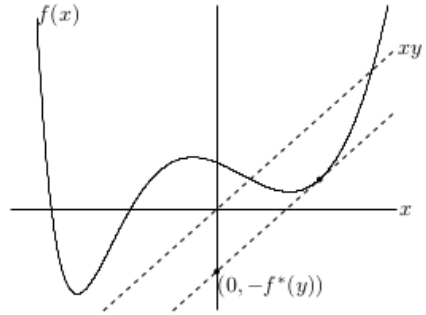
Conjugate

Let $f : \mathbb{R}^n \to \mathbb{R} \bigcup \{\infty\}$ be a convex function. The conjugate of f is

$$f^*(y) = \sup_x (y^\intercal x - f(x))$$

Note

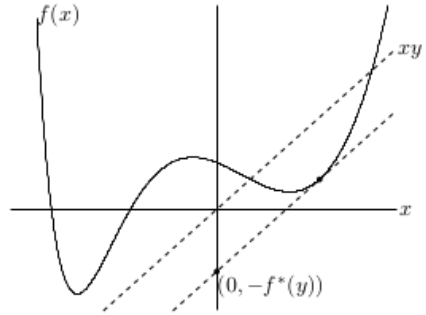
- $f^*(y)$ is the minimal value of β such that the affine function $y^T x - \beta$ underestimates $f(x)$ everywhere.
- Equivalently, the distance we need to lower the hyperplane $y^\intercal x - t = 0$ in order to get a supporting hyperplane to $\mathbf{epi}\, f$.
- $y^\intercal x - t = f^*(y)$ are the supporting hyperplanes of $\mathbf{epi}\, f$



$$f^*(y) = \sup_x (y^\intercal x - f(x))$$

Properties of the Conjugate

1. $f^{**} = f$
2. f^* is a convex function



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Properties of the Conjugate

1. $f^{**} = f$
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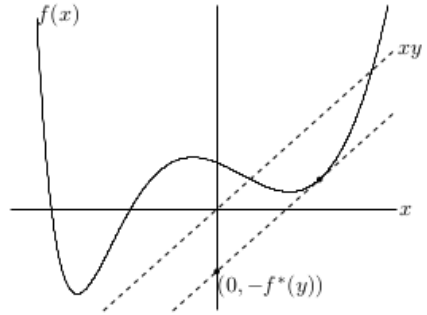
2. Supremum of affine functions of y



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Properties of the Conjugate

1. $f^{**} = f$
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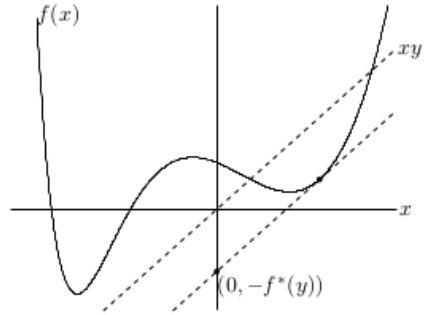
1.
   - $f^{**}(x) = \max_y y^\intercal x - f^*(y)$ when f is convex
   - For fixed y, $f^*(y)$ is minimal β such that $y^\intercal x - \beta$ underestimates f.
   - Therefore $f^{**}(x)$ is the maximum, over all y, of affine underestimates $y^\intercal x - \beta$ of f
   - By our characterization early in this lecture, this is equal to f.



$$f^*(y) = \sup_x (y^\intercal x - f(x))$$

Properties of the Conjugate

1. $f^{**} = f$
2. f^* is a convex function

Note

When f is non-convex, $f^* = (convexhull(f))^*$, and $f^{**} = convexhull(f)$.

Examples

- Affine function: $f(x) = ax + b$. Conjugate has $f^*(a) = b$, and ∞ elsewhere
- $f(x) = x^2/2$ is self-conjugate
- Exponential: $f(x) = e^x$. Conjugate has $f^*(y) = y \log y - y$ for $y \in \mathbb{R}_+$, and ∞ elsewhere.
- Quadratic: $f(x) = \frac{1}{2} x^\intercal Q x$ with $Q \succeq 0$. Self conjugate.
- Log-sum-exp: $f(x) = \log(\sum_i e^{x_i})$. Conjugate has $f^*(y) = \sum_i y_i \log y_i$ for $y \succeq 0$ and $1^\intercal y = 1$, ∞ otherwise.