CS672: Approximation Algorithms
Spring 14
Introduction to Linear Programming I

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Outline

1. Linear Programming
2. Application to Combinatorial Problems
3. Duality and Its Interpretations
4. Properties of Duals

Outline

A Brief History

- The forefather of convex optimization problems, and the most ubiquitous.
  - Best understood in that context
  - But this is not a convex optimization class
- Developed by Kantorovich during World War II (1939) for planning the Soviet army's expenditures and returns. Kept secret.
- Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs
- John von Neumann developed LP duality in 1947, and applied it to game theory
- Polynomial-time solvable under fairly general conditions
  - Ellipsoid method (Khachiyan 1979)
  - Interior point methods (Karmarkar 1984).

LP General Form

$$
\begin{array}{lll}
\text{minimize (or maximize)} & c^\intercal x & \\
\text{subject to} & a_i^\intercal x \leq b_i, & \text{for } i \in \mathcal{C}^1. \\
& a_i^\intercal x \geq b_i, & \text{for } i \in \mathcal{C}^2. \\
& a_i^\intercal x = b_i, & \text{for } i \in \mathcal{C}^3.
\end{array}
$$

- Decision variables: $x \in \mathbb{R}^n$
- Parameters:
  - $c \in \mathbb{R}^n$ defines the linear objective function
  - $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ define the i'th constraint

Standard Forms

Packing Form

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

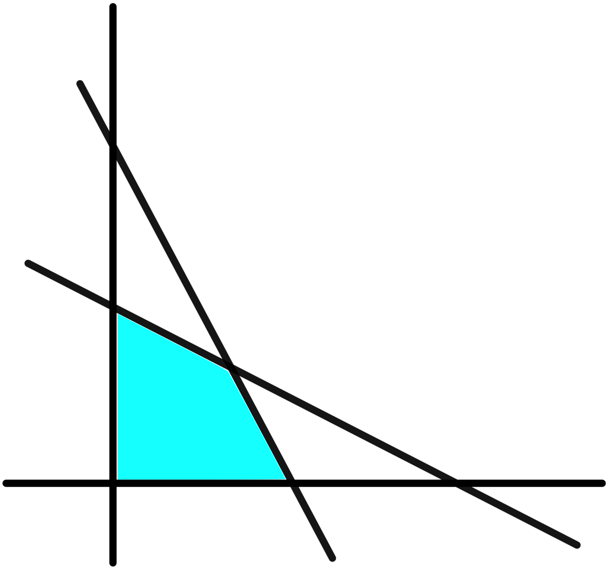
Covering Form

$$\begin{aligned} \text{minimize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \succeq b \\ & x \succeq 0 \end{aligned}$$

Every LP can be transformed to either form

- minimizing $c^\intercal x$ is equivalent to maximizing $-c^\intercal x$
- inequality constraints can be flipped by multiplying by -1
- Each equality constraint can be replaced by two inequalities
- Uconstrained variable x_j can be replaced by $x_j^+ - x_j^-$, where both x_j^+ and x_j^- are constrained to be nonnegative.

Geometric View



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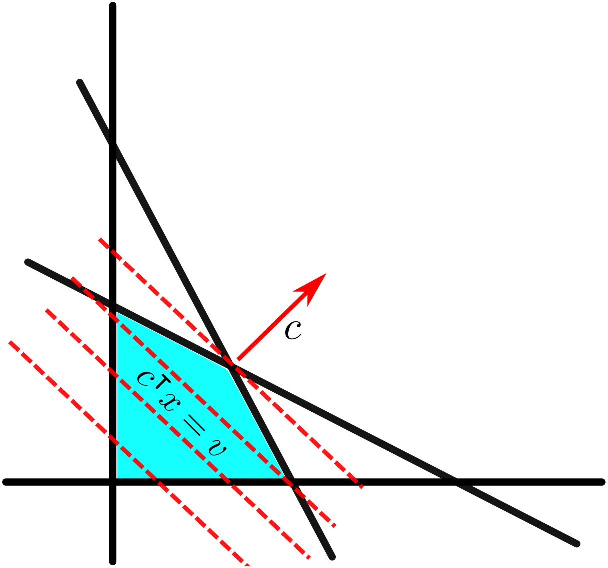

A 2-D example

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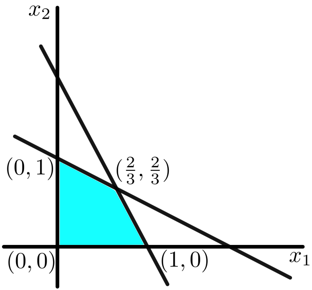

Economic Interpretation: Optimal Production

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Facility wants to maximize profit subject to available raw materials

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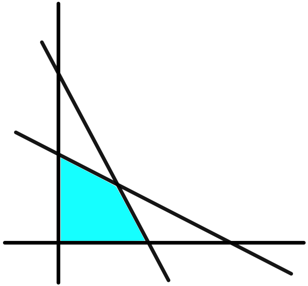
\begin{array}{lll}
\text{maximize} & c^\intercal x & \\
\text{subject to} & a_i^\intercal x \leq b_i, & \text{for } i = 1, \ldots, m. \\
 & x_j \geq 0, & \text{for } j = 1, \ldots, n.
\end{array}
$$

Terminology

- Hyperplane: The region defined by a linear equality
- Halfspace: The region defined by a linear inequality $a_i^\intercal x \leq b_i$.
- Polyhedron: The intersection of a set of linear inequalities in Euclidean space
  - Feasible region of an LP is a polyhedron
- Polytope: A bounded polyhedron
  - Equivalently: convex hull of a finite set of points
- Vertex: A point x is a vertex of polyhedron P if $\nexists y \neq 0$ with $x + y \in P$ and $x - y \in P$
- Face of P: The intersection with P of a hyperplane H disjoint from the interior of P



Basic Facts about LPs and Polytopes

Fact

Feasible regions of LPs (i.e. polyhedrons) are convex

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Set of optimal solutions of an LP is convex

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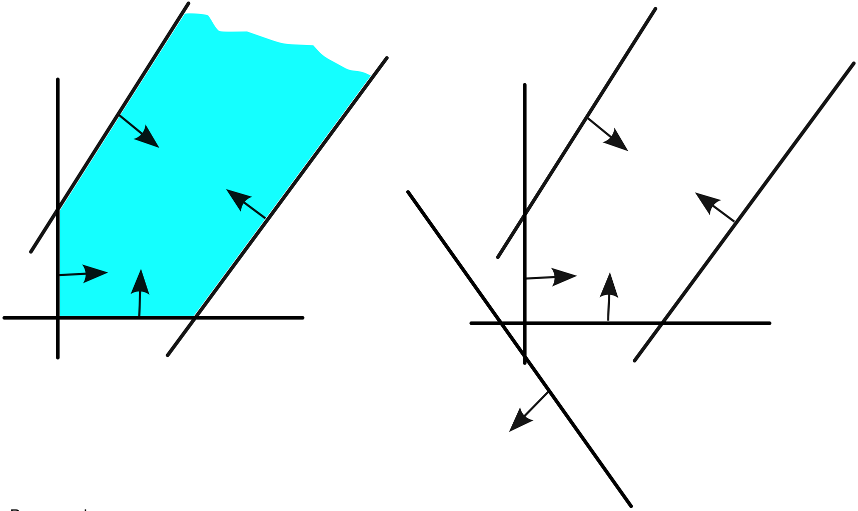
Fact

At a vertex, n linearly independent constraints are satisfied with equality (a.k.a. tight)

Basic Facts about LPs and Polyhedrons

Fact

An LP either has an optimal solution, or is unbounded or infeasible



Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

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Proof

- Assume not, and take a non-vertex optimal solution x with the maximum number of tight constraints
- There is $y \neq 0$ s.t. $x \pm y$ are feasible
- y is perpendicular to the objective function and the tight constraints at x.
  - i.e. $c^\intercal y = 0$, and $a_i^\intercal y = 0$ whenever the i'th constraint is tight for x.
- Can choose y s.t. $y_j < 0$ for some j
- Let α be the largest constant such that $x + \alpha y$ is feasible
  - Such an α exists
- An additional constraint becomes tight at $x + \alpha y$, a contradiction.

Counting non-zero Variables

Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most m non-zero variables.

$$\begin{array}{lll} \text{maximize} & c^\intercal x & \\ \text{subject to} & a_i^\intercal x \le b_i, & \text{for } i = 1, \ldots, m. \\ & x_j \ge 0, & \text{for } j = 1, \ldots, n. \end{array}$$

- e.g. for optimal production with n products and m raw materials, there is an optimal plan with at most m products.

Outline

Application to Combinatorial Problems

- Linear programs often encode combinatorial problems either exactly or approximately
- Since our focus is on NP-hard problems, we encounter mostly the latter
  - An LP often relaxes the problem
  - Allows "better than optimal" solutions which are fractional

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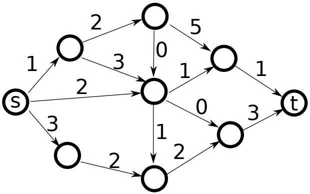
Uses

1. Rounding a solution of the LP
2. Analysis via primal/dual paradigm

Example: Shortest Path

Given a directed network $G = (V, E)$ where edge e has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from s to t.



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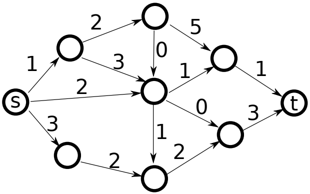

Shortest Path LP

$$
\begin{array}{lll}
\text{minimize} & \sum_{e \in E} \ell_e x_e & \\
\text{subject to} & \sum\limits_{e \to v} x_e - \sum\limits_{v \to e} x_e = \delta_v, & \text{for } v \in V. \\
& x_e \geq 0, & \text{for } e \in E.
\end{array}
$$

Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

Example: Vertex Cover

Given an undirected graph $G = (V, E)$, with weights w_i for $i \in V$, find minimum-weight $S \subseteq V$ “covering” all edges.

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Vertex Cover LP

$$
\begin{array}{lll}
\text{minimize} & \sum_{i \in V} w_i x_i & \\
\text{subject to} & x_i + x_j \geq 1, & \text{for } (i, j) \in E. \\
 & x_i \geq 0, & \text{for } i \in V.
\end{array}
$$

Example: Knapsack

Given n items with sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$, and a knapsack of capacity C, find the maximum value set of items which fits in the knapsack.

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Knapsack LP

$$
\begin{array}{lll}
\text{maximize} & \sum_{i=1}^{n} v_i x_i & \\
\text{subject to} & \sum_{i=1}^{n} s_i x_i \leq C & \\
 & x_i \leq 1, & \text{for } i \in [n]. \\
 & x_i \geq 0, & \text{for } i \in [n].
\end{array}
$$

Outline

Linear Programming Duality

Primal LP

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \end{array}$$

Dual LP

$$\begin{array}{ll} \text{minimize} & b^\intercal y \\ \text{subject to} & A^\intercal y = c \\ & y \succeq 0 \end{array}$$

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- y_i is the dual variable corresponding to primal constraint $A_i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint corresponding to primal variable x_j

Linear Programming Duality: Standard Form, and Visualization

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Interpretation 1: Economic Interpretation

Recall the Optimal Production problem

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- Dual variable y_i is a proposed price per unit of raw material i
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials

Interpretation 2: Finding the Best Upperbound

Recall the simple LP

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.

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- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$, with an optimal value of $4/3$.
- What if, instead of finding the optimal solution, we saught to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by $\frac{1}{3}$ and summing gives $x_1 + x_2 \leq 4/3$.

Interpretation 2: Finding the Best Upperbound

	x_1	x_2	x_3	x_4	
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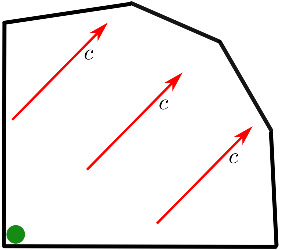
- Multiplying each row i by y_i and summing gives the inequality

$$y^T Ax \leq y^T b$$

- When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$.
- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside polytope $Ax \leq b$.

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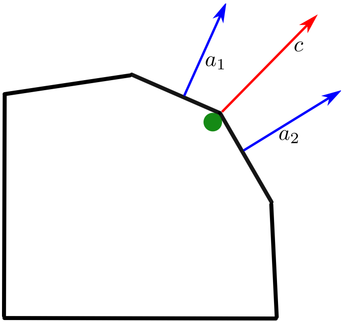

- Apply force field c to a ball inside polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.

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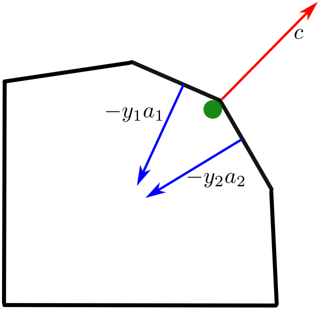

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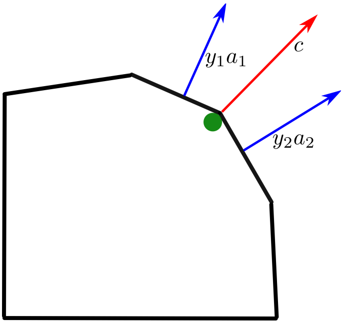

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- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

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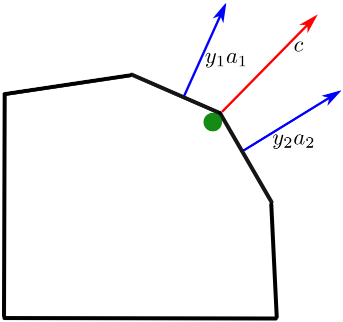

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- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- Dual can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)

Outline

Duality is an Inversion

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \succeq c \\ & y \succeq 0 \end{aligned}$$

Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.

Correspondance Between Variables and Constraints

Primal LP

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & & \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\ & x_j \geq 0, & \text{for } j \in [n]. \end{array}$$

Dual LP

$$\begin{array}{lll} \min & \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & & \\ & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, & \text{for } j \in [n]. \\ & y_i \geq 0, & \text{for } i \in [m]. \end{array}$$

Correspondance Between Variables and Constraints

Primal LP

$$
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\text{s.t.} & & \\
{\color{red} y_i :} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\
 & x_j \geq 0, & \text{for } j \in [n].
\end{array}
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Dual LP

$$
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- The i'th primal constraint gives rise to the i'th dual variable y_i

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- The i'th primal constraint gives rise to the i'th dual variable y_i
- The j'th primal variable x_j gives rise to the j'th dual constraint

Syntactic Rules

Primal LP

$$
\begin{array}{lll}
\text{max} & c^\intercal x & \\
\text{s.t.} & & \\
y_i : & a_i x \leq b_i, & \text{for } i \in \mathcal{C}_1. \\
y_i : & a_i x = b_i, & \text{for } i \in \mathcal{C}_2. \\
 & x_j \geq 0, & \text{for } j \in \mathcal{D}_1. \\
 & x_j \in \mathbb{R}, & \text{for } j \in \mathcal{D}_2.
\end{array}
$$

Dual LP

$$
\begin{array}{lll}
\text{min} & b^\intercal y & \\
\text{s.t.} & & \\
x_j : & \overline{a}_j^\intercal y \geq c_j, & \text{for } j \in \mathcal{D}_1. \\
x_j : & \overline{a}_j^\intercal y = c_j, & \text{for } j \in \mathcal{D}_2. \\
 & y_i \geq 0, & \text{for } i \in \mathcal{C}_1. \\
 & y_i \in \mathbb{R}, & \text{for } i \in \mathcal{C}_2.
\end{array}
$$

Rules of Thumb

- Loose constraint (i.e. inequality) $\Rightarrow$ tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) $\Rightarrow$ loose dual variable (i.e. unconstrained)