CS672: Approximation Algorithms
Spring 14
Introduction to Linear Programming II

Instructor: Shaddin Dughmi

Outline

1. Recall: Duality and Its Interpretations
2. Weak and Strong Duality
3. Consequences of Duality
4. Uses and Examples of Duality
5. Solvability of LP

Outline

Linear Programming Duality

"Flexible" Form:

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y = c \\ & y \succeq 0 \end{aligned}$$

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Packing/Covering form:

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Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \succeq c \\ & y \succeq 0 \end{aligned}$$

Interpretation 1: Economic Interpretation

Primal LP

max $\sum_{j=1}^{n} c_j x_j$
s.t.
$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \forall i \in [m].$
$x_j \geq 0, \quad \forall j \in [n].$

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Facility wants to maximize profit

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Dual LP

min $\sum_{i=1}^{m} b_i y_i$
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- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Facility wants to maximize profit

- y_i is a proposed price per unit of raw material i
- Feasibility means facility has incentive to sell as opposed to produce
- Buyer wants to spend as little as possible to buy materials

Interpretation 2: Finding the Best Upperbound

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^{\intercal} x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^{\intercal} y \\ \text{subject to} \quad & A^{\intercal} y \succeq c \\ & y \succeq 0 \end{aligned}$$

- Multiplying each row i by y_i and summing gives the inequality

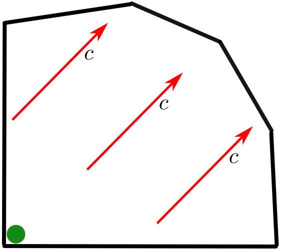
$$y^T Ax \leq y^T b$$

- When $y^T A \succeq c^T$, we have

$$c^T x \leq y^T Ax \leq y^T b$$

- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved by combining inequalities this way.

Interpretation 3: Physical Forces



- Apply force field c to a ball inside polytope $Ax \preceq b$.

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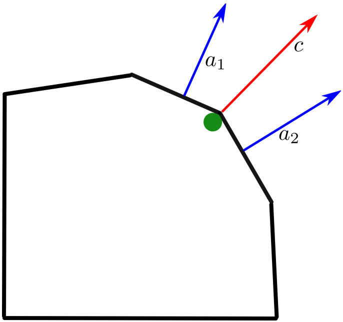

- Apply force field c to a ball inside polytope $Ax \preceq b$.
- Eventually, ball will come to rest against the walls of the polytope.

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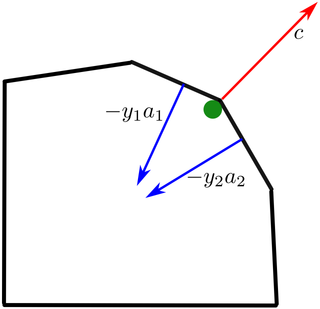

- Apply force field c to a ball inside polytope $Ax \preceq b$.
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- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball

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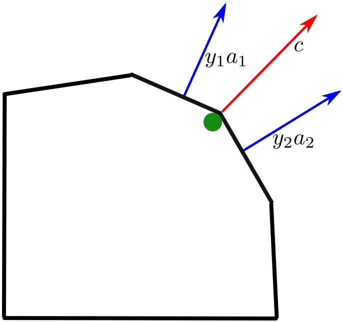

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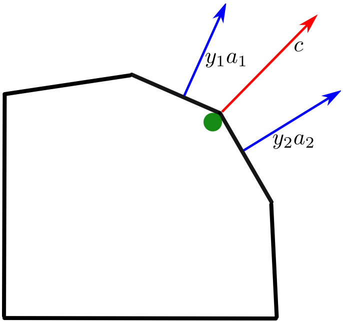
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

Interpretation 3: Physical Forces



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- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- Dual can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring ball back to origin by moving polytope

Outline

Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \succeq c \\ & y \succeq 0 \end{aligned}$$

Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \leq b^\intercal y$.

Corollary

- *If primal and dual both feasible and bounded,* $OPT(Primal) \leq OPT(Dual)$
- *If primal is unbounded, dual is infeasible*
- *If dual is unbounded, primal is infeasible*

Weak Duality

Primal LP

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Theorem (Weak Duality)

For every primal feasible x and dual feasible y, we have $c^\intercal x \leq b^\intercal y$.

Corollary

If x is primal feasible, and y is dual feasible, and $c^\intercal x = b^\intercal y$, then both are optimal.

Interpretation of Weak Duality

Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

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Upperbound Interpretation

Self explanatory

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Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

Upperbound Interpretation

Self explanatory

Physical Interpretation

Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.

Proof of Weak Duality

Primal LP

$$\begin{aligned} \text{maximize} \quad & c^\intercal x \\ \text{subject to} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & b^\intercal y \\ \text{subject to} \quad & A^\intercal y \succeq c \\ & y \succeq 0 \end{aligned}$$

$$c^\intercal x \leq y^\intercal A x \leq y^\intercal b$$

Strong Duality

Primal LP

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Dual LP

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Theorem (Strong Duality)

If either the primal or dual is feasible and bounded, then so is the other and $OPT(Primal) = OPT(Dual)$.

Interpretation of Strong Duality

Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

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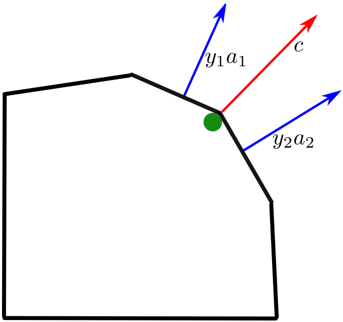
Upperbound Interpretation

The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.

Physical Interpretation

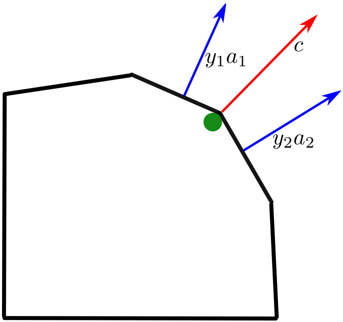
There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.

Informal Proof of Strong Duality



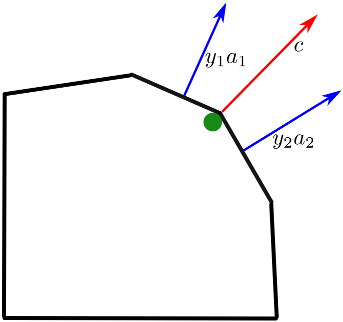
- Recall the physical interpretation of duality

Informal Proof of Strong Duality



- Recall the physical interpretation of duality
- When ball is stationary at x, we expect force c to be neutralized only by constraints that are tight
  - i.e. force multipliers y such that $y_i(b_i - a_i x) = 0$

Informal Proof of Strong Duality



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$$y^\intercal b - c^\intercal x = y^\intercal b - y^T Ax = \sum_i y_i(b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!

Outline

Complementary Slackness

Primal LP

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- Let $s_i = (b - Ax)_i$ be the i'th primal slack variable
- Let $t_j = (A^\intercal y - c)_j$ be the j'th dual slack variable

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Complementary Slackness

x and y are optimal if and only if

- $x_j t_j = 0$ for all $j = 1, \ldots, n$
- $y_i s_i = 0$ for all $i = 1, \ldots, m$

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Interpretation of Complementary Slackness

Economic Interpretation

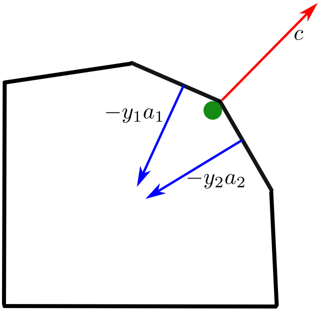
Given an optimal primal production vector x and optimal dual offer prices y,

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than 0

Interpretation of Complementary Slackness

Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.



Proof of Complementary Slackness

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Proof of Complementary Slackness

Primal LP

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Dual LP

$$\begin{array}{ll} \text{minimize} & y^\intercal b \\ \text{subject to} & A^\intercal y - t = c \\ & y \succeq 0 \\ & t \succeq 0 \end{array}$$

- Can equivalently rewrite LP using slack variables

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$$y^\intercal b - c^\intercal x = y^\intercal (Ax + s) - (y^\intercal A - t^\intercal)x = y^\intercal s + t^\intercal x$$

Proof of Complementary Slackness

Primal LP

maximize	$c^\intercal x$
subject to	$Ax + s = b$
	$x \succeq 0$
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Dual LP

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Gap between primal and dual objectives is 0 if and only if complementary slackness holds.

Recovering Primal from Dual

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Primal LP
(n variables, $m + n$ constraints)

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  - Exactly n dual constraints are loose

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- n loose dual constraints impose n tight primal constraints

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  - Exactly m of the $m + n$ dual constraints are tight at y
  - Exactly n dual constraints are loose
- n loose dual constraints impose n tight primal constraints
  - Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution x.

Outline

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3. Consequences of Duality
4. **Uses and Examples of Duality**
5. Solvability of LP

Uses of Duality in Algorithm Design

1. Gain structural insights
   - Dual of a problem gives a “different way of looking at it”.
2. As a benchmark; i.e. to certify (approximate) optimality
   - The primal/dual paradigm
   - A dual may be explicitly constructed by the algorithm, or as part of its analysis

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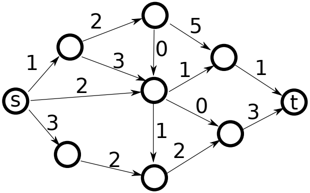
   - A dual may be explicitly constructed by the algorithm, or as part of its analysis

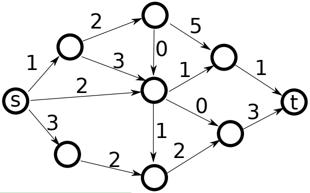
Let’s look at some duals and interpret them.

Shortest Path

Given a directed network $G = (V, E)$ where edge e has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from s to t.



Shortest Path



Primal LP

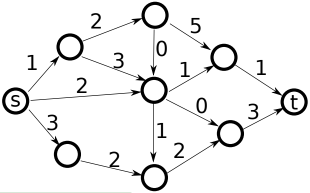
min $\sum_{e \in E} \ell_e x_e$
s.t.
$$\sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V.$$
$$x_e \geq 0, \quad \forall e \in E.$$

Dual LP

max $y_t - y_s$
s.t.
$$y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E.$$

Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

Shortest Path



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s.t.

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Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

Interpretation of Dual

Stretch s and t as far apart as possible, subject to edge lengths.

Vertex Cover

Given an undirected graph $G = (V, E)$, with weights w_i for $i \in V$, find minimum-weight $S \subseteq V$ "covering" all edges.

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Primal LP

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s.t.

$x_i + x_j \geq 1, \qquad \forall (i,j) \in E.$

$x \succeq 0$

Dual LP

max $\sum_{e \in E} y_e$

s.t.

$$\sum_{e \in \Gamma(i)} y_e \leq w_i, \qquad \forall i \in V.$$

$y \succeq 0$

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& x \succeq 0
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\text{max} \quad & \textstyle\sum_{e \in E} y_e \\
\text{s.t.} \quad & \\
& \sum_{e \in \Gamma(i)} y_e \leq w_i, \qquad \forall i \in V. \\
& y \succeq 0
\end{aligned}
$$

Interpretation of Dual

Trying to "sell" coverage to edges at prices y_e.

- Objective: Maximize revenue
- Feasible: charge any neighborhood (of a vertex i) no more than it would cost them if they broke away and bought i themselves

Outline

Solvability of Explicit Linear Programs

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

- In the examples we have seen so far, the linear program is explicit
- I.e. the constraint matrix A, as well as rhs vector b and objective c, are of polynomial size.

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- I.e. the constraint matrix A, as well as rhs vector b and objective c, are of polynomial size.

Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

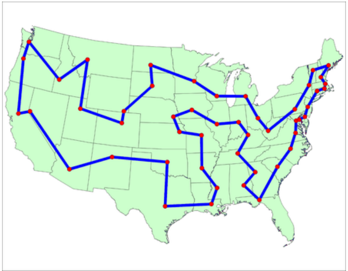
Originally using the ellipsoid algorithm, and more recently interior-point algorithms which are more efficient in practice.

Implicit Linear Programs

- These are linear programs in which the number of constraints is exponential (in the natural description of the input)
- These are useful as an analytical tool
- Can be solved in many cases!

Implicit Linear Programs

- These are linear programs in which the number of constraints is exponential (in the natural description of the input)
- These are useful as an analytical tool
- Can be solved in many cases!
- E.g. Held-Karp relaxation for TSP



$$\begin{aligned} \text{min} \quad & \textstyle\sum_{e \in E} d_e x_e \\ \text{s.t.} \quad & \\ & x(\delta(S)) \geq 2, \qquad \forall \emptyset \subset S \subset V. \\ & x(\delta(v)) = 2, \qquad \forall v \in V. \\ & 0 \preceq x \preceq 1 \end{aligned}$$

Where $\delta(S)$ denotes the edges going out of $S \subseteq V$.

Solvability of Implicit Linear Programs

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

Theorem (Polynomial Solvability of Implicit LP)

Consider a family Π of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

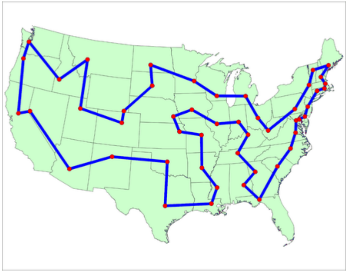
- *A separation oracle for the polyhedron $Ax \preceq b$*
- *Explicit access to c*

Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ are at most $\text{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for Π (both primal and dual).

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $Ax \preceq b$ or finds a violated constraint $a_i x > b_i$.

E.g. of a Separation Oracle



$$\begin{aligned}
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&\text{s.t.} \\
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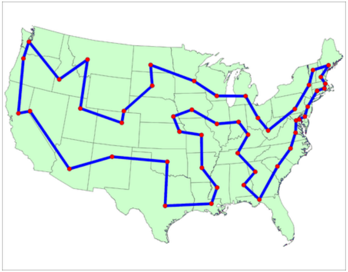

min $\sum_{e \in E} d_e x_e$

s.t.

$x(\delta(S)) \geq 2, \qquad \forall \emptyset \subset S \subset V.$

$x(\delta(v)) = 2, \qquad \forall v \in V.$

$0 \preceq x \preceq 1$

- Nontrivial part: given fixed x need to check whether $x(\delta(S)) \geq 2$ for all S, else find such an S which violates this.

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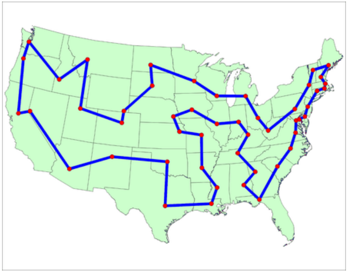

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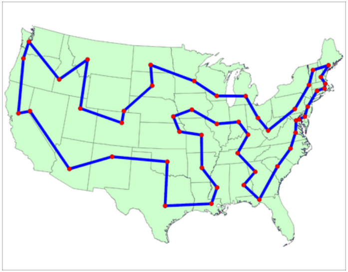

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- This is min-cut in a weighted graph, which we can solve.