CS675: Convex and Combinatorial Optimization
Fall 2014
Combinatorial Problems as Convex Programs

Instructor: Shaddin Dughmi

The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of V into $(S, V \setminus S)$ maximizing number of edges with exactly one end in S.

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-x_i x_j}{2} & \\ \text{subject to} & x_i \in \{-1, 1\}, & \text{for } i \in V. \end{array}$$

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Instead of requiring x_i to be on the 1 dimensional sphere, we relax and permit it to be in the n-dimensional sphere.

Vector Program relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-\vec{v}_i \cdot \vec{v}_j}{2} & \\ \text{subject to} & ||\vec{v}_i||_2 = 1, & \text{for } i \in V. \\ & \vec{v}_i \in \mathbb{R}^n, & \text{for } i \in V. \end{array}$$

SDP Relaxation

- Recall: An $n \times n$ matrix Y is PSD iff $Y = V^T V$ for $n \times n$ matrix V
- When diagonal entires of Y are 1, V has unit length columns
- Equivalently: PSD matrices encode pairwise dot products of columns of V
- Recall: Y and V can be recovered from each other efficiently

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$$\begin{array}{lll} \text{maximize} & \sum_{(i,j)\in E} \frac{1-Y_{ij}}{2} & \\ \text{subject to} & Y_{ii} = 1, & \text{for } i \in V. \\ & Y \in S_+^n & \end{array}$$

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Randomized Algorithm for Max Cut

1. Solve the SDP to get $Y \succeq 0$
2. Decompose Y to VV^T
3. Pick a random vector r on the unit sphere
4. Place all nodes i with $v_i \cdot r \geq 0$ on one side of the cut, and all others on the other side

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Lemma

The SDP cuts each edge with probability at least $0.878\frac{1-Y_{ij}}{2}$

Consequently, by linearity of expectation, expected number of edges cut is at least $0.878\, OPT$.

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We use the following fact

Fact

For all angles $\theta \in [0, \pi]$,

$$\frac{\theta}{\pi} \geq 0.878 \cdot \frac{1}{2}(1 - cos(\theta))$$

to prove the Lemma on the board.