CS675: Convex and Combinatorial Optimization
Fall 2014
Duality of Convex Optimization Problems

Instructor: Shaddin Dughmi

Outline

Recall: Optimization Problem in Standard Form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
& h_i(x) = 0, \quad \text{for } i = 1, \ldots, k.
\end{array}
$$

- For convex optimization problems in standard form, f_i is convex and h_i is affine.
- Let $\mathcal{D}$ denote the domain of all these functions (i.e. when their value is finite)

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This Lecture + Next

We will develop duality theory for convex optimization problems, generalizing linear programming duality.

Running Example: Linear Programming

We have already seen the standard form LP below

$$\begin{array}{ll} \text{maximize} & c^\intercal x \\ \text{subject to} & Ax \preceq b \\ & x \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & -c^\intercal x \\ \text{subject to} & Ax - b \preceq 0 \\ & -x \preceq 0 \end{array}$$

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Along the way, we will recover the following standard form dual

$$\begin{array}{ll} \text{minimize} & y^\intercal b \\ \text{subject to} & A^\intercal y \succeq c \\ & y \succeq 0 \end{array}$$

The Lagrangian

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Basic idea of Lagrangian duality is to relax/soften the constraints by replacing each with a linear “penalty term” or “cost” in the objective.

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The Lagrangian Function

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x)$$

- λ_i is Lagrange Multiplier for i'th inequality constraint
  - Required to be nonnegative
- ν_i is Lagrange Multiplier for i'th equality constraint
  - Allowed to be of arbitrary sign

The Lagrange Dual Function

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, & \text{for } i = 1, \ldots, k. \end{array}$$

The Lagrange dual function gives the optimal value of the primal problem subject to the softened constraints

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The Lagrange Dual Function

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)$$

- Observe: g is a concave function of the Lagrange multipliers
- We will see: Its quite common for the Lagrange dual to be unbounded $(-\infty)$ for some λ and ν
- By convention, domain of g is (λ, ν) s.t. $g(\lambda, \nu) > -\infty$

Langrange Dual of LP

$$\begin{aligned} \text{minimize} \quad & -c^\intercal x \\ \text{subject to} \quad & Ax - b \preceq 0 \\ & -x \preceq 0 \end{aligned}$$

First, the Lagrangian function

$$\begin{aligned} L(x, \lambda) &= -c^\intercal x + \lambda_1^\intercal (Ax - b) - \lambda_2^\intercal x \\ &= (A^\intercal \lambda_1 - c - \lambda_2)^\intercal x - \lambda_1^\intercal b \end{aligned}$$

Langrange Dual of LP

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And the Lagrange Dual

$$
\begin{aligned}
g(\lambda) &= \inf_x L(x, \lambda) \\
&= \begin{cases} -\infty & \text{if } A^\mathsf{T}\lambda_1 - c - \lambda_2 \neq 0 \\ -\lambda_1^\mathsf{T} b & A^\mathsf{T}\lambda_1 - c - \lambda_2 = 0 \end{cases}
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So we restrict the domain of g to λ satisfying $A^\intercal \lambda_1 - c - \lambda_2 = 0$

Interpretation: "Soft" Lower Bound

$$\begin{array}{lll} \text{min} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & \text{for } i = 1, \ldots, m. \\ & h_i(x) = 0, & \text{for } i = 1, \ldots, k. \end{array}$$

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Fact

$g(\lambda, \nu)$ is a lowerbound on OPT(primal) for every $\lambda \succeq 0$ and $\nu \in \mathbb{R}^k$.

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Proof

- Every primal feasible x incurs nonpositive penalty by $L(x, \lambda, \nu)$
- Therefore, $L(x^*, \lambda, \nu) \le f_0(x^*)$
- So $g(\lambda, \nu) \le f_0(x^*) = OPT(Primal)$

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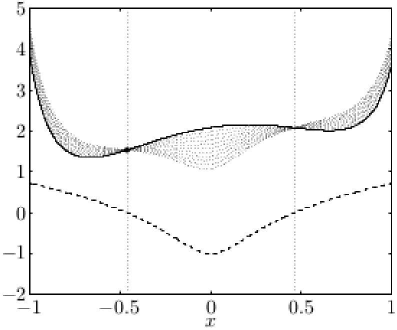
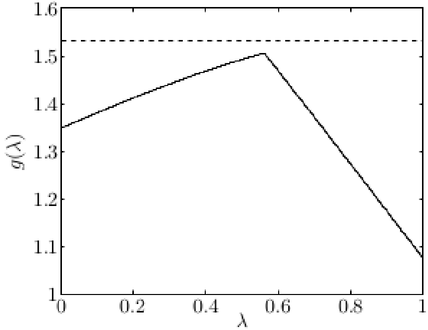
Interpretation

- A "hard" feasibility constraint can be thought of as imposing a penalty of $+\infty$ if violated
- Lagrangian imposes a "soft" linear penalty for violating a constraint, and a reward for slack
- Lagrange dual finds the optimal subject to these soft constraints

Interpretation: “Soft” Lower Bound

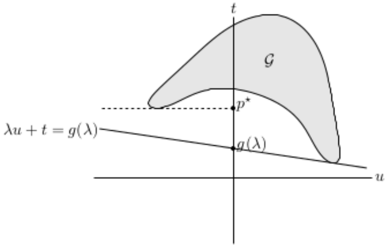




Interpretation: Geometric

Most easily visualized in the presence of a single inequality constraint

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0 \end{array}$$



- Let $\mathcal{G}$ be attainable constraint/objective function value tuples
  - i.e. $(u, t) \in \mathcal{G}$ if there is an x such that $f_1(x) = u$ and $f_0(x) = t$
- $p^* = \inf \{t : (u, t) \in \mathcal{G}, u \leq 0\}$
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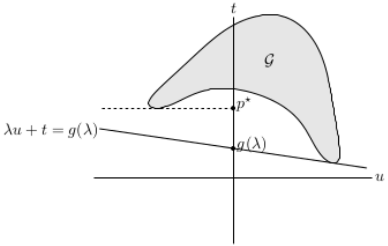

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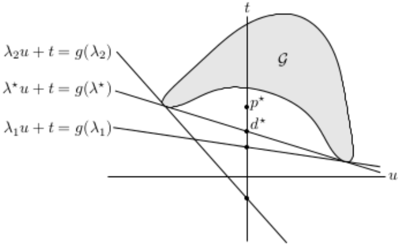
- $\lambda u + t = g(\lambda)$ is a supporting hyperplane to $\mathcal{G}$ pointing northeast
- Must intersect vertical axis below p^*
- Therefore $g(\lambda) \leq p^*$

This is the problem of finding the best lower bound on OPT(primal) implied by the Lagrange dual function

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$



- Note: this is a convex optimization problem, regardless of whether primal problem was convex
- By convention, sometimes we add "dual feasibility" constraints to impose "nontrivial" lowerbounds (i.e. $g(\lambda, \nu) \geq -\infty$)
- (λ^*, ν^*) solving the above are referred to as the dual optimal solution

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Recall

Our Lagrange dual function for the above LP (to the right), defined over the domain $A^\intercal \lambda_1 - c - \lambda_2 = 0$.

$$g(\lambda) = -\lambda_1^\intercal b$$

Langrange Dual Problem of LP

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The Lagrange dual problem can then be written as

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Another Example: Conic Optimization Problem

$$\begin{array}{ll} \text{minimize} & c^\intercal x \\ \text{subject to} & Ax = b \\ & x \in K \end{array}$$

- $x \in K$ can equivalently be written as $z^\intercal x \leq 0$, $\forall z \in K^\circ$

$$\begin{aligned} L(x, \lambda, \nu) &= c^\intercal x + \nu^\intercal (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^\intercal x \\ &= (c - A^\intercal \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^\intercal x + \nu^\intercal b \end{aligned}$$

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- Can think of $\lambda \succeq 0$ as choosing some $s \in K^\circ$

$$L(x, s, \nu) = (c - A^\intercal \nu + s)^\intercal x + \nu^\intercal b$$

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Outline

Weak Duality

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min $f_0(x)$
s.t.
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Dual Problem

max $g(\lambda, \nu)$
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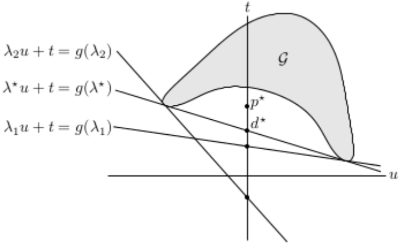
Weak Duality

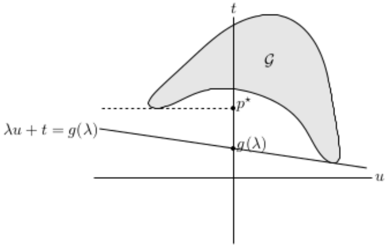
$OPT(dual) \leq OPT(primal)$.



- We have already argued holds for every optimization problem
- Duality Gap: difference between optimal dual and primal values

Recall: Geometric Interpretation of Weak Duality

minimize $\quad f_0(x)$
subject to $\quad f_1(x) \le 0$



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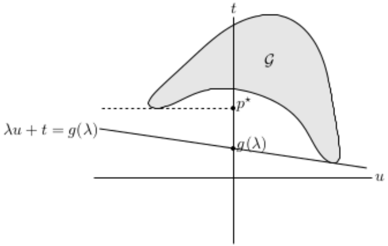

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Fact

The equation $\lambda u + t = g(\lambda)$ defines a supporting hyperplane to $\mathcal{G}$, intersecting t axis at $g(\lambda) \leq p^*$.

Strong Duality

Strong Duality

We say strong duality holds if $OPT(dual) = OPT(primal)$.

- Equivalently: there exists a setting of Lagrange multipliers so that $g(\lambda, \nu)$ gives a tight lowerbound on primal optimal value.
- In general, does not hold for non-convex optimization problems
- Usually, but not always, holds for convex optimization problems.
  - Mild assumptions, such as Slater's condition, needed.

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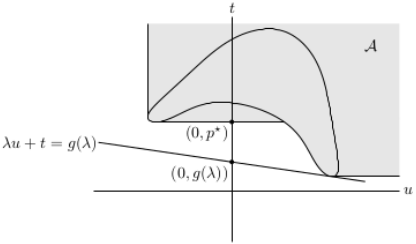

- Let $\mathcal{A}$ be everything northeast (i.e. “worse”) than $\mathcal{G}$
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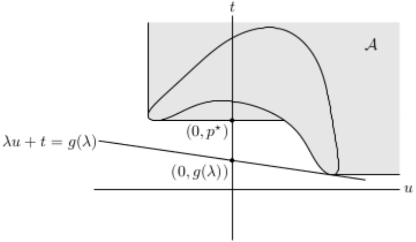
Geometric Proof of Strong Duality

minimize	$f_0(x)$
subject to	$f_1(x) \leq 0$



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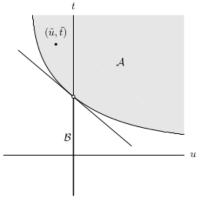

Fact

When f_0 and f_1 are convex, $\mathcal{A}$ is convex.

Geometric Proof of Strong Duality

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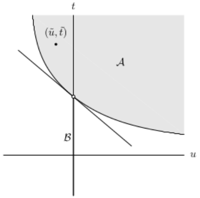

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When f_0 and f_1 are convex, $\mathcal{A}$ is convex.

Proof

- Assume (u, t) and (u', t') are in $\mathcal{A}$

Geometric Proof of Strong Duality

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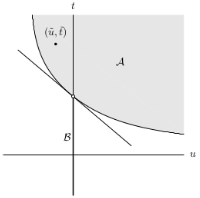

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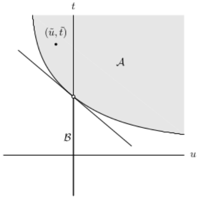

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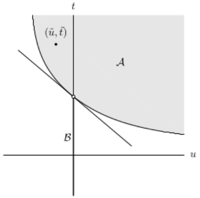

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  $(f_1(\alpha x + (1-\alpha)x'), f_0(\alpha x + (1-\alpha)x')) \leq (\alpha u + (1-\alpha)u', \alpha t + (1-\alpha)t')$
- Therefore, midpoint of (u, t) and (u', t') also in $\mathcal{A}$.

Geometric Proof of Strong Duality

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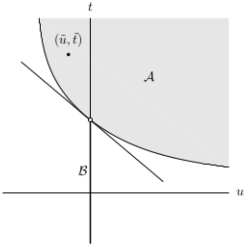

Theorem (Informal)

There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

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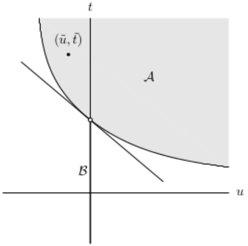


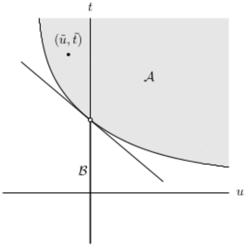
Theorem (Informal)

There is a choice of λ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

Proof

- Recall $(0, p^*)$ is on the boundary of $\mathcal{A}$
- By the supporting hyperplane theorem, there is a supporting hyperplane to $\mathcal{A}$ at $(0, p^*)$
- Direction of the supporting hyperplane gives us an appropriate λ

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \le 0 \end{array}$$



- In our proof, we ignored a technicality that can prevent strong duality from holding.

I Lied (A little)

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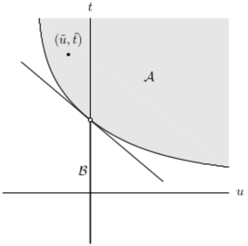

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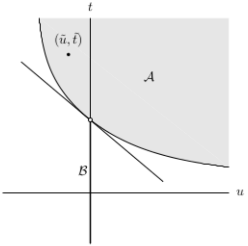

- In our proof, we ignored a technicality that can prevent strong duality from holding.
- If our supporting hyperplane H at $(0, p^*)$ is vertical, then no finite λ exists such that $(\lambda, 1)$ is normal to H.
- Somewhat counterintuitively, this can happen even in simple convex optimization problems (though its somewhat rare in practice)

Violation of Strong Duality

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & \frac{x^2}{y} \leq 0 \end{array}$$

- Let domain of constraint be region $y \geq 1$
- Problem is convex, with feasible region given by $x = 0$
- Optimal value is 1, at $x = 0$ and $y = 1$

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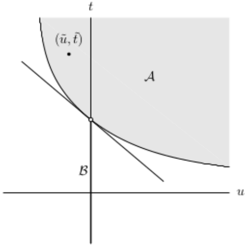
- Let domain of constraint be region $y \geq 1$
- Problem is convex, with feasible region given by $x = 0$
- Optimal value is 1, at $x = 0$ and $y = 1$
- $\mathcal{A} = \mathbb{R}^2_{++} \bigcup (\{0\} \times [1, \infty])$
- Therefore, any supporting hyperplane to $\mathcal{A}$ at $(0, 1)$ must be vertical.

Slater's Condition

There exists a point $x \in \mathcal{D}$ where all inequality constraints are strictly satisfied (i.e. $f_i(x) < 0$). I.e. the optimization problem is strictly feasible.



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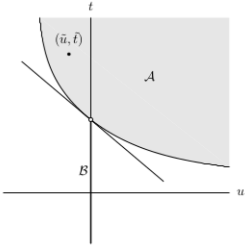

- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical
- Can be weakened to requiring strict feasibility only of non-affine constraints