

CS675: Convex and Combinatorial Optimization
Fall 2014
The Ellipsoid Algorithm

Instructor: Shaddin Dughmi

History and Basics

- Originally developed in the mid 70s by Iudin, Nemirovski, and Shor for use as a heuristic for convex optimization
- In 1979, Kachiyan shows that it yields a polynomial time algorithm for LP

History and Basics

- Originally developed in the mid 70s by Iudin, Nemirovski, and Shor for use as a heuristic for convex optimization
- In 1979, Kachiyan shows that it yields a polynomial time algorithm for LP
- Inefficient in practice, has been surpassed by newer interior point methods.

History and Basics

- Originally developed in the mid 70s by Iudin, Nemirovski, and Shor for use as a heuristic for convex optimization
- In 1979, Kachiyan shows that it yields a polynomial time algorithm for LP
- Inefficient in practice, has been surpassed by newer interior point methods.
- Theoretically most powerful, with deep consequences for complexity and optimization
 - Polynomial-time algorithm for linear programming
 - Polynomial-time algorithm for approximate convex optimization under mild conditions
 - Equivalence of separation and optimization

Outline

- 1 Description of The Ellipsoid Method
- 2 Properties

Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing K
- A rational number $R > 0$ satisfying $\text{vol}(E) \leq R$.
- A rational number $r > 0$ such that if K is nonempty, then $\text{vol}(K) \geq r$.

Find a point $x \in K$ or declare that K is empty.

- Recall: convex optimization reduces to checking feasibility
 - We will explain later how the parameters r, R, E figure into this
- Description of K : any description that admits an efficient implementation of a **separation oracle**

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

Examples:

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

Examples:

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of A corresponding to a constraint violated by x .

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

Examples:

- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of A corresponding to a constraint violated by x .
- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \nabla f_i(x)$ for some violated constraint.

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

Examples:

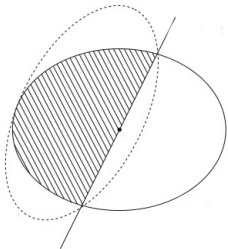
- Explicitly written polytope $Ay \leq b$: take $h = a_i$ to the row of A corresponding to a constraint violated by x .
- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \nabla f_i(x)$ for some violated constraint.
- The positive semi-definite cone S_n^+ : Let H be the outer product vv^\top of an eigenvector v of X corresponding to a negative eigenvalue.



- Unit ball $B(0, 1) = \{x \in \mathbb{R}^n : x^\top x \leq 1\}$.
- **Ellipsoid**: the result of applying some affine transformation $x \rightarrow Lx + c$ to $B(0, 1)$.
 - Gives the set $\{x : (x - c)^\top L^{-\top} L^{-1} (x - c) \leq 1\}$
- It is conventional to take $Q = LL^\top$, which is symmetric PSD, and write the ellipsoid as

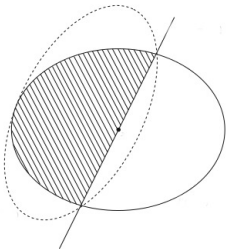
$$E(c, Q) = \{x : (x - c)^\top Q^{-1} (x - c) \leq 1\}$$

- Can calculate **volume** easily: $\text{vol}(E(c, Q)) = \sqrt{\det Q} \text{vol}(B(0, 1))$
- Note: since $Q \succeq 0$ has a PSD square root, can take L to be PSD as well



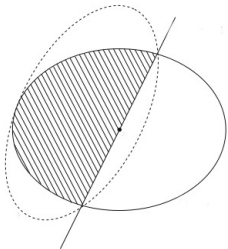
Ellipsoid Method

- 1 Start with initial ellipsoid $E = E(c, Q) \supseteq K$
- 2 Using the separation oracle, check if the center $c \in K$.
 - If so, terminate and output c .
 - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \cap \{y : h^\top y \leq h^\top c\}$
- 3 Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
- 4 If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.



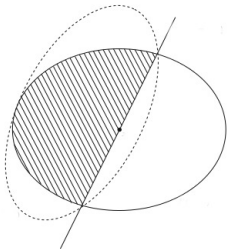
Ellipsoid Method

- 1 Start with initial ellipsoid $E = E(c, Q) \supseteq K$
- 2 **Using the separation oracle, check if the center $c \in K$.**
 - If so, terminate and output c .
 - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \cap \{y : h^\top y \leq h^\top c\}$
- 3 Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
- 4 If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.



Ellipsoid Method

- 1 Start with initial ellipsoid $E = E(c, Q) \supseteq K$
- 2 Using the separation oracle, check if the center $c \in K$.
 - If so, terminate and output c .
 - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \cap \{y : h^\top y \leq h^\top c\}$
- 3 **Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.**
- 4 If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.



Ellipsoid Method

- 1 Start with initial ellipsoid $E = E(c, Q) \supseteq K$
- 2 Using the separation oracle, check if the center $c \in K$.
 - If so, terminate and output c .
 - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \cap \{y : h^\top y \leq h^\top c\}$
- 3 Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
- 4 If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.

Outline

- 1 Description of The Ellipsoid Method
- 2 Properties

Theorem

If the ellipsoid algorithm terminates, then it either outputs $x \in K$ or correctly declares that K is empty.

- Algorithm outputs some x only if it was certified in K by the separation oracle
- $E \supseteq K$ is maintained throughout the algorithm, by definition
- We are promised $\text{vol}(K) < r$ only if $K = \emptyset$
- If algorithm outputs empty then $r > \text{vol}(E) \geq \text{vol}(K)$, and $K = \emptyset$.

To prove that the ellipsoid method runs in polynomial time, we need the following lemma

Lemma

- The minimum volume ellipsoid E' containing half of another ellipsoid E can be computed in $\text{poly}(n)$ operations.
- Moreover, the volume of E' is smaller than the volume of E by a factor of at least $e^{\frac{1}{2(n+1)}}$.

This is tricky to prove, so we will come back to it later.

Theorem

The ellipsoid method terminates after $2(n + 1) \ln \frac{R}{r}$ iterations. Moreover, each iteration can be implemented using $\text{poly}(n)$ operations and a single call to the separation oracle.

- Volume of working ellipsoid starts off at R
- Decreases by a factor of $e^{\frac{1}{2(n+1)}}$ each iteration.
- After $2(n + 1) \ln \frac{R}{r}$ iterations, volume is less than r , triggering termination.

Theorem

The ellipsoid method terminates after $2(n + 1) \ln \frac{R}{r}$ iterations. Moreover, each iteration can be implemented using $\text{poly}(n)$ operations and a single call to the separation oracle.

- Volume of working ellipsoid starts off at R
- Decreases by a factor of $e^{\frac{1}{2(n+1)}}$ each iteration.
- After $2(n + 1) \ln \frac{R}{r}$ iterations, volume is less than r , triggering termination.
- In each iteration, we
 - Call the separation oracle (Assumed constant time)
 - Compute the minimum volume ellipsoid ($\text{poly}(n)$ time by lemma)
 - Compute the volume of E' (reduces to determinant computation).

Lemma

- The minimum volume ellipsoid E' containing half of another ellipsoid E can be computed in $\text{poly}(n)$ operations.
- Moreover, the volume of E' is smaller than the volume of E by a factor of at least $e^{\frac{1}{2(n+1)}}$.

Proof on board.