

CS675: Convex and Combinatorial Optimization
Fall 2014
Consequences of the Ellipsoid Algorithm

Instructor: Shaddin Dughmi

Outline

- 1 Recapping the Ellipsoid Method
- 2 Complexity of Convex Optimization
- 3 Complexity of Linear Programming
- 4 Equivalence of Separation and Optimization

Recall: Feasibility Problem

The ellipsoid method solves the following problem.

Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing K
- A rational number $R > 0$ satisfying $\text{vol}(E) \leq R$.
- A rational number $r > 0$ such that if K is nonempty, then $\text{vol}(K) \geq r$.

Find a point $x \in K$ or declare that K is empty.

- Equivalent variant: drop the requirement on volume $\text{vol}(K)$, and either find a point $x \in K$ or an ellipsoid $E \supseteq K$ with $\text{vol}(E) < r$.

All the ellipsoid method needed was the following subroutine

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separating x from K .

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\top x \geq h^\top y$ for all $y \in K$.
- Equivalently, K is contained in the halfspace

$$H(h, x) = \{y : h^\top y \leq h^\top x\}$$

with x at its boundary.

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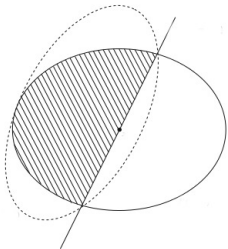
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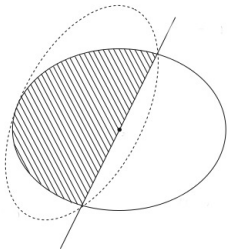
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- The positive semi-definite cone S_n^+ : Let H be the outer product vv^\top of an eigenvector v of X corresponding to a negative eigenvalue.



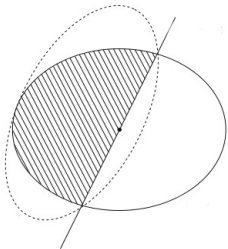
Ellipsoid Method

- 1 Start with initial ellipsoid $E = E(c, Q) \supseteq K$
- 2 Using the separation oracle, check if the center $c \in K$.
 - If so, terminate and output c .
 - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \cap \{y : h^\top y \leq h^\top c\}$
- 3 Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
- 4 If $\text{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return “empty”.



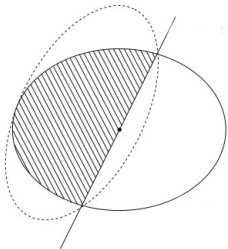
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Using T to denote the runtime of the separation oracle

Theorem

The ellipsoid algorithm terminates in time polynomial n , $\ln \frac{R}{r}$, and T , and either outputs $x \in K$ or correctly declares that K is empty.

We proved most of this. For the rest, see references.

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Note

For runtime polynomial in input size we need

- T polynomial in input size
- $\frac{R}{r}$ exponential in input size

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Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex

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- Recall: A problem Π is a family of **instances** $I = (f, \mathcal{X})$
- When represented explicitly, often given in **standard form**

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad \text{for } i \in \mathcal{C}_1. \\ & a_i^\top x = b_i, \quad \text{for } i \in \mathcal{C}_2. \end{array}$$

- The functions $f, \{g_i\}_i$ are given in some parametric form allowing evaluation of each function and its derivatives.

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- We will abstract away details of how instances of a problem are represented, but denote the length of the description by $\langle I \rangle$
- We simply require polynomial time (in $\langle I \rangle$ and n) separation oracles and such.

Solvability of Convex Optimization

There are many subtly different “solvability statements”. This one is the most useful, yet simple to describe, IMO.

Requirements

We say an algorithm **weakly solves** a convex optimization problem in **polynomial time** if it:

- Takes an approximation parameter $\epsilon > 0$
- Terminates in time $\text{poly}(\langle I \rangle, n, \log(\frac{1}{\epsilon}))$
- Returns an **ϵ -optimal** $x \in \mathcal{X}$:

$$f(x) \leq \min_{y \in \mathcal{X}} f(y) + \epsilon [\max_{y \in \mathcal{X}} f(y) - \min_{y \in \mathcal{X}} f(y)]$$

Theorem (Polynomial Solvability of CP)

Consider a family Π of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

- A **separation oracle** for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$
- A **first order oracle** for f : evaluates $f(x)$ and $\nabla f(x)$.
- An algorithm which **computes a starting ellipsoid** $E \supseteq \mathcal{X}$ with $\frac{\text{vol}(E)}{\text{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.

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Let's now prove this, by reducing to the ellipsoid method

Simplifying Assumption

Assume we are given $\min_{y \in \mathcal{X}} f(y)$ and $\max_{y \in \mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

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We can feed this into the Ellipsoid method!

Needed Ingredients

- 1 Separation oracle for new feasible set K :
- 2 Ellipsoid E containing K :
- 3 Guarantee that $\frac{\text{vol}(E)}{\text{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

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$$K = \{x \in \mathcal{X} : f(x) \leq \epsilon\}$$

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$$\text{vol}(K) \geq \epsilon^n \text{vol}(X).$$

This shows that $\text{vol}(K)$ is only exponentially smaller (in n and $\log \frac{1}{\epsilon}$) than $\text{vol}(\mathcal{X})$, and therefore also $\text{vol}(E)$, so it suffices.

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- Let $y = \epsilon x$ for $x \in \mathcal{X}$, and invoke Jensen's inequality

$$f(y) = f(\epsilon x + (1 - \epsilon)0) \leq \epsilon f(x) + (1 - \epsilon)f(0) \leq \epsilon$$

Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
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- We don't need to know anything about T !

Key Observation

We don't really need to know T , H , or L to simulate the same execution of the ellipsoid method on K !!

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find x
subject to $x \in \mathcal{X}$
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- Require separation oracle for K to use ∇f only as a last resort
 - This is allowed.
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- Run this simulation until enough iterations have passed, and take the best feasible point encountered. This must be in K .

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- In the implicit case, we require polynomial time in the bit complexity of individual entries of A , b , c .

Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \leq b$: trivial when explicitly represented
- A first order oracle for $c^T x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
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Solution to both issues involves tedious accounting of numerical issues

Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

Lemma

Let v be vertex of the polyhedron $Ax \leq b$. It is the case that v has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.

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- Bounding ellipsoid: all vertices contained in the box $-2^M \leq x \leq 2^M$, which in turn is contained in an ellipsoid of volume exponential in M and n .

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Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Volume lowerbound when feasible set is full dimensional: follows from bit complexity of vertices.
- More generally, need to instead solve a “relaxed problem”. Specifically, relaxing to $Ax \leq b + \epsilon$, for sufficiently small ϵ with $\langle \epsilon \rangle = \text{poly}(M)$. Gives volume exponentially small in M , but no smaller. Still close enough to original polyhedron so solution to relaxed problem can be “rounded” to solution of the latter.

Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

Lemma

Let v be vertex of the polyhedron $Ax \leq b$. It is the case that v has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Rounding to a vertex: If a point y is ϵ -optimal, for sufficiently small ϵ chosen carefully to polynomial in description of input, then rounding to the nearest x with M bits recovers the vertex.

Theorem (Polynomial Solvability of Implicit LP)

Consider a family Π of linear programming problems $I = (A, b, c)$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

- A **separation oracle** for the polyhedron $Ax \leq b$
- Explicit access to c

Moreover, assume that every $\langle a_{ij} \rangle, \langle b_i \rangle, \langle c_j \rangle$ are at most $\text{poly}(\langle I \rangle, n)$. Then there is a polynomial time algorithm for Π (both primal and dual).

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 - It turns out this is still OK, but takes a lot of work.

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For the dual, we need equivalence of separation and optimization (HW).

Outline

- 1 Recapping the Ellipsoid Method
- 2 Complexity of Convex Optimization
- 3 Complexity of Linear Programming
- 4 Equivalence of Separation and Optimization**

Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle
- As it turns out, the two tasks are polynomial-time equivalent.

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Lets formalize the two questions, parametrized by a polytope P .

Linear Optimization Problem

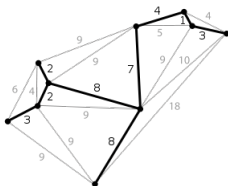
- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $\operatorname{argmax}_{x \in P} c^\top x$.

Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P$, or else find $h \in \mathbb{R}^n$ s.t. $h^\top x < h^\top y$ for all $x \in P$.

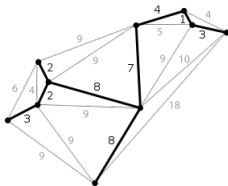
Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs c_e on edges e , find a minimum cost spanning tree of G .



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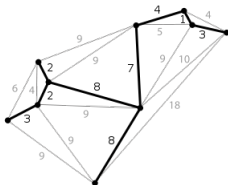


Spanning Tree Polytope

$$\begin{aligned} \sum_{e \subseteq X} x_e &\leq |X| - 1, & \text{for } X \subset V. \\ \sum_{e \in E} x_e &= n - 1 \\ x_e &\geq 0, & \text{for } e \in E. \end{aligned}$$

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- Optimization: Find the minimum/maximum weight spanning tree
- Separation: Find $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
 - i.e. When edge weights are x , find a “dense” subgraph

Theorem (Equivalence of Separation and Optimization for Polytopes)

Consider a family \mathcal{P} of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \text{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\text{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\text{poly}(\langle P \rangle, n, \langle c \rangle)$ time.

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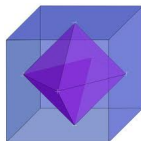
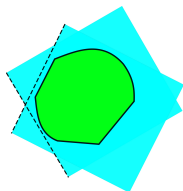
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- We already sketched the proof of the forward direction
 - Separation \Rightarrow optimization
- For the other direction, we need **polars**

Recall: Polar Duality of Convex Sets



One way of representing the all halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The **polar** of S is defined as follows:

$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $a^\top x \leq b$ with $b \neq 0$ can be written as a “normalized” inequality $y^\top x \leq 1$, by dividing by b .
- S° can be thought of as the normalized representations of halfspaces containing S .

Properties of the Polar

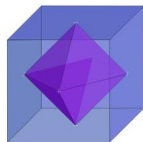
- 1 If S is bounded and $0 \in \text{interior}(S)$, then the same holds for S° .
- 2 $S^{\circ\circ} = S$



$$S = \{x : y \cdot x \leq 1 \text{ for all } y \in S^\circ\}$$



$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$



Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A .

- Facets of P correspond to vertices of P° .
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Proof Outline: Optimization \Rightarrow Separation

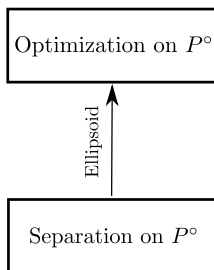
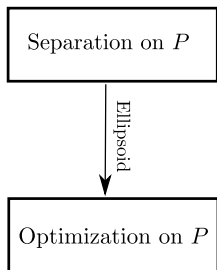


Separation on P

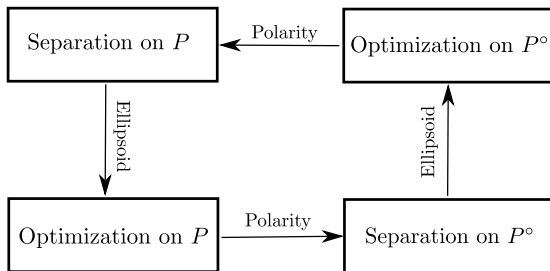
Ellipsoid

Optimization on P

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Separation over S reduces in constant time to optimization over S° , and vice versa since $S^{\circ\circ} = S$.



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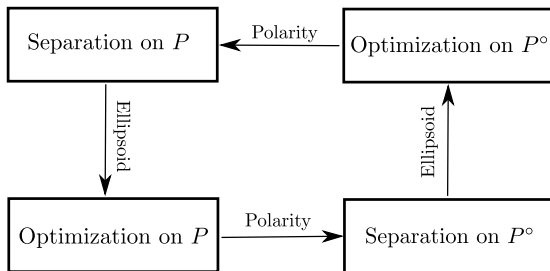
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- equivalently, iff $\max_{y \in S^\circ} y \cdot x \leq 1$.
- If we find $y \in S^\circ$ s.t. $y \cdot x > 1$, then y is the separating hyperplane
 - $y^\top z \leq 1 < y^\top x$ for every $z \in S$.

Optimization \iff Separation



Beyond Polytopes

Essentially everything we proved about equivalence of separation and optimization for polytopes extends to (approximately) to arbitrary convex sets.

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Problems parametrized by P , a closed convex set.

Weak Optimization Problem

- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $x \in P^{+\epsilon}$, and $c^\top x \geq \max_{x' \in P} c^\top x' - \epsilon$

Weak Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P^{-\epsilon}$, or else find $h \in \mathbb{R}^n$ with $\|h\| = 1$ s.t. $h^\top x < h^\top y + \epsilon$ for all $x \in P$.

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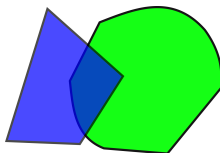
I could have equivalently stated the weak optimization problem for convex functions instead of linear.

Theorem (Equivalence of Separation and Optimization for Convex Sets)

Consider a family \mathcal{P} of convex sets described implicitly using $\langle P \rangle$ bits. Then the weak separation problem is solvable in $\text{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the weak optimization problem is also solvable in $\text{poly}(\langle P \rangle, n, \langle c \rangle)$ time.

- The “approximation” in this statement is necessary, since we can’t solve convex optimization problems exactly.
- Weak separation suffices for ellipsoid, which is only approximately optimal anyways
- By polarity, weak optimization is equivalent to weak separation

Implication: Operations preserving solvability

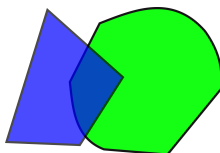


- Assume you can efficiently optimize over two convex sets P and Q

Question

What about $P \cup Q$ and $P \cap Q$?

Implication: Operations preserving solvability



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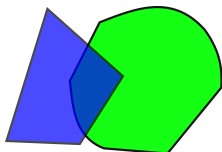
Question

What about $P \cup Q$ and $P \cap Q$?

$P \cup Q$

- Yes! Simply optimize over each separately, and take the better of the two outcomes.
- Equivalent to optimizing over the convex hull of $P \cup Q$.
- Implication of Separation/optimization equivalence: there is a separation oracle for $\text{convexhull}(P \cup Q)$.

Implication: Operations preserving solvability



- Assume you can efficiently optimize over two convex sets P and Q

Question

What about $P \cup Q$ and $P \cap Q$?

$P \cap Q$

- Yes! Follows from equivalence of separation and optimization.
- Specifically, can separate over P and Q individually, therefore can separate over $P \cap Q$, and then can optimize over $P \cap Q$.
- Applications: colorful spanning tree, cardinality-constrained matching, ...

Problem

Given a point $x \in \mathcal{P}$, where $\mathcal{P} \subseteq \mathbb{R}^n$ is a solvable polytope, write x as a convex combination of $n + 1$ vertices of \mathcal{P} , and do so in polynomial time.

- Existence: Caratheodory's theorem.
- E.g. Birkhoff Von-Neumann, fractional spanning trees, fractional matchings, ...
- Follows from equivalence of separation and optimization. Two very different yet instructive proofs, see HW4.