CS675: Convex and Combinatorial Optimization
Fall 2014
Optimality Conditions for Convex Optimization

Instructor: Shaddin Dughmi

Outline

Recall: Lagrangian Duality

Primal Problem

min $f_0(x)$
s.t.
$f_i(x) \leq 0, \quad \forall i = 1, \ldots, m.$
$h_i(x) = 0, \quad \forall i = 1, \ldots, k.$

Dual Problem

max $g(\lambda, \nu)$
s.t.
$\lambda \succeq 0$

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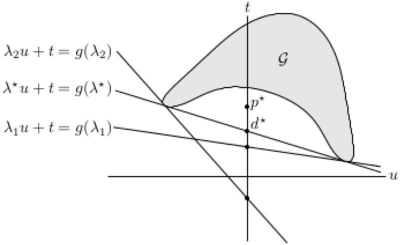
Weak Duality

$OPT(dual) \leq OPT(primal)$.



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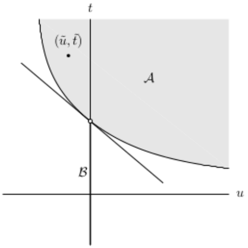

Strong Duality

$OPT(dual) = OPT(primal)$.

Dual Solution as a Certificate

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- Dual solutions serves as a certificate of optimality
- If $f_0(x) = g(\lambda, \nu)$, and both are feasible, then both are optimal.

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- If $f_0(x) = g(\lambda, \nu)$, and both are feasible, then both are optimal.
- If $f_0(x) - g(\lambda, \nu) \leq \epsilon$, then both are within ϵ of optimality.
    - OPT(primal) and OPT(dual) lie in the interval $[g(\lambda, \nu), f_0(x)]$

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  - OPT(primal) and OPT(dual) lie in the interval $[g(\lambda, \nu), f_0(x)]$
- Primal-dual algorithms use dual certificates to recognize optimality, or bound sub-optimality.

Complementary Slackness

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Facts

If strong duality holds, and x^* and (λ^*, ν^*) are optimal, then

- x^* minimizes $L(x, \lambda^*, \nu^*)$ over all x.
- $\lambda_i^* f_i(x^*) = 0$ for all i. (Complementary Slackness)

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Proof

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, \nu^*) \\
&\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{k} \nu_i^* h_i(x^*) \\
&\leq f_0(x^*)
\end{aligned}
$$

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Interpretation

- Lagrange multipliers (λ^*, ν^*) “simulate” the primal feasibility constraints
- Interpreting λ_i as the “value” of the i'th constraint, at optimality only the binding constraints are “valuable”
  - Recall economic interpretation of LP

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KKT Conditions

When strong duality holds, the primal problem is convex, and the constraint functions are differentiable, x^* and (λ^*, ν^*) are optimal iff:

- x^* and (λ^*, ν^*) are feasible
- $\lambda_i^* f_i(x^*) = 0$ (Complementary Slackness)
- $\bigtriangledown_x L(x^*, \lambda^*, \nu^*) = \bigtriangledown f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \bigtriangledown f_i(x^*) + \sum_{i=1}^{k} \nu_i^* \bigtriangledown h_i(x^*) = 0$

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Why are KKT Conditions Useful?

- Derive an analytical solution to some convex optimization problems
- Gain structural insights

Example: Equality-constrained Quadratic Program

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}x^\intercal Px + q^\intercal x + r \\
\text{subject to} & Ax = b
\end{array}
$$

- KKT Conditions: $Ax^* = b$ and $Px^* + q + A^\intercal \nu^* = 0$
- Simply a solution of a linear system with variables x^* and ν^*.

Example: Market Equilibria (Fisher's Model)

- Buyers B, and goods G.
- Buyer i has utility u_{ij} for each unit of good G.
- Buyer i has budget m_i, and there's one divisible unit of each good.

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- Does there exist a market equilibrium?
  - Prices p_j on items, such that each player can buy his favorite bundle and the market clears.

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Eisenberg-Gale Convex Program

$$
\begin{array}{lll}
\text{maximize} & \sum_i m_i \log \sum_j u_{ij} x_{ij} & \\
\text{subject to} & \sum_i x_{ij} \le 1, & \text{for } j \in G. \\
 & x \succeq 0 &
\end{array}
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Using KKT conditions, we can prove that the dual variables corresponding to the item supply constraints are market-clearing prices!