CS675: Convex and Combinatorial Optimization
Fall 2014
Submodular Function Optimization

Instructor: Shaddin Dughmi

Outline

Introduction

- We saw how matroids form a class of feasible sets over which optimization of modular objectives is tractable
- If matroids are discrete analogues of convex sets, then submodular functions are discrete analogues of convex/concave functions
  - Submodular functions behave like convex functions sometimes (minimization) and concave other times (maximization)
- Today we will introduce submodular functions, go through some examples, and mention some of their properties

Set Functions

- A set function takes as input a set, and outputs a real number
  - Inputs are subsets of some ground set X
  - $f : 2^X \to \mathbb{R}$
- We will focus on set functions where X is finite, and denote $n = |X|$

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- We will focus on set functions where X is finite, and denote $n = |X|$
- Equivalently: map points in the hypercube $\{0, 1\}^n$ to the real numbers
  - Can be plotted as 2^n points in $n + 1$ dimensional space

Set Functions

- We have already seen modular set functions
  - Associate a weight w_i with each $i \in X$, and set $f(S) = \sum_{i \in S} w_i$
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- Supmodular/supermodular functions are weak analogues to convex/concave functions (in no particular order!)
- Other possibly useful properties a set function may have:
  - Monotone increasing or decreasing
  - Nonnegative: $f(A) \geq 0$ for all $S \subseteq X$
  - Normalized: $f(\emptyset) = 0$.

Submodular Functions

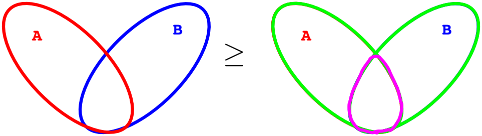
Definition 1

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

for all $A, B \subseteq X$.

- "Uncrossing" two sets reduces their total function value



Submodular Functions

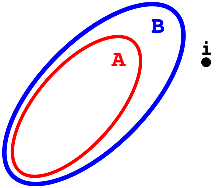
Definition 2

A set function $f : 2^X \to \mathbb{R}$ is submodular if and only if

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$

for all $A \subseteq B \subseteq X$ and $i \notin B$.

- The marginal value of an additional element exhibits "diminishing marginal returns"
- Should remind of concavity



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Examples

Many common examples are monotone, normalized, and submodular. We mention some.

Coverage Functions

X is the left hand side of a graph, and $f(S)$ is the total number of neighbors of S.

- Can think of $i \in X$ as a set, and $f(S)$ as the total "coverage" of S.

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Probability

X is a set of probability events, and $f(S)$ is the probability at least one of them occurs.

Examples

Social Influence

- X is the family of nodes in a social network
- A meme, idea, or product is adopted at a set of nodes S
- The idea propagates through the network through some random diffusion process
    - Many different models
- $f(S)$ is the expected number of nodes in the network which end up adopting the idea.

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Utility Functions

When X is a set of goods, $f(S)$ can represent the utility of an agent for a bundle of these goods. Utilities which exhibit diminishing marginal returns are natural in many settings.

Examples

Entropy

X is a set of random variables, and $f(S)$ is the entropy of the joint distribution of a subset of them S.

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Clustering Quality

X is the set of nodes in a graph G, and $f(S) = E(S)$ is the internal connectedness of cluster S.

- Supermodular

Examples

There are fewer examples of non-monotone submodular/supermodular functions, which are nontheless fundamental.

Graph Cuts

X is the set of nodes in a graph G, and $f(S)$ is the number of edges crossing the cut $(S, X \setminus S)$.

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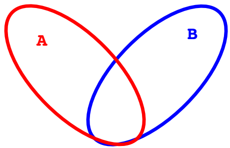
- However, maximizing it reduces to maximizing supermodular function $E(S) - \alpha|S|$ for various $\alpha > 0$ (binary search)

Equivalence of Both Definitions



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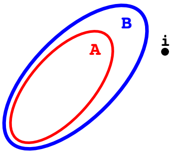

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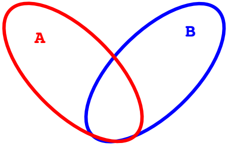

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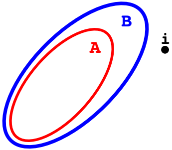

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$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A))$$

Definition 1 $\Rightarrow$ Definition 2

- To prove (2), let $A' = A \bigcup \{i\}$ and $B' = B$ and apply (1)

$$\begin{aligned} f(A \cup \{i\}) + f(B) &= f(A') + f(B') \\ &\geq f(A' \cap B') + f(A' \cup B') \\ &= f(A) + f(B \cup \{i\}) \end{aligned}$$

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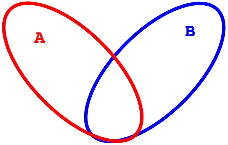

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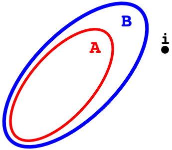

Definition 2 $\Rightarrow$ Definition 1

- To prove (1), start with $A = B$ and repeatedly elements to one but not the other
- At each step, (2) implies that the LHS of inequality (1) increases more than the RHS

Operations Preserving Submodularity

- Nonnegative-weighted combinations (a.k.a. conic combinations): If $f_1, \ldots, f_k$ are submodular, and $w_1, \ldots, w_k \geq 0$, then $g(S) = \sum_i w_i f_i(S)$ is also submodular
  - Special case: adding or subtracting a modular function

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Note

The minimum or maximum of two submodular functions is not necessarily submodular

Optimizing Submodular Functions

- As our examples suggest, optimization problems involving submodular functions are very common
- These can be classified on two axes: constrained/unconstrained and maximization/minimization

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) $O(1)$ ("nice" constriants)	Usually NP-hard to apx. Few easy special cases

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$$\begin{aligned} \text{minimize} \quad & f(S) \\ \text{subject to} \quad & S \subseteq X \end{aligned}$$

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An algorithm which runs in time polynomial in n and b.

Note: weakly polynomial. There are strongly polytime algorithms.

Examples

Minimum Cut

Given a graph $G = (V, E)$, find a set $S \subseteq V$ minimizing the number of edges crossing the cut $(S, V \setminus S)$.

- G may be directed or undirected.
- Extends to hypergraphs.

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Densest Subgraph

Given an undirected graph $G = (V, E)$, find a set $S \subseteq V$ maximizing the average internal degree.

- Reduces to supermodular maximization via binary search for the right density.

Continuous Extensions of a Set Function

Recall

A set function f on $X = \{1, \ldots, n\}$ with can be thought of as a map from the vertices $\{0, 1\}^n$ of the n-dimensional hypercube to the real numbers.

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We will consider extensions of a set function to the entire hypercube.

Extension of a Set Function

Given a set function $f : \{0, 1\}^n \to \mathbb{R}$, an extension of f to the hypercube $[0, 1]^n$ is a function $g : [0, 1]^n \to \mathbb{R}$ satisfying $g(x) = f(x)$ for every $x \in \{0, 1\}^n$.

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Long story short...

We will exhibit an extension which is convex when f is submodular, and can be minimized efficiently. We will then show that minimizing it yields a solution to the submodular minimization problem.

The Convex Closure

Convex Closure

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, the convex closure $f^- : [0,1]^n \to \mathbb{R}$ of f is the point-wise greatest convex function under-estimating f on $\{0,1\}^n$.

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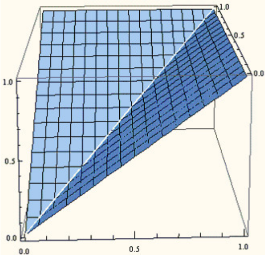
Geometric Intuition

What you would get by placing a blanket under the plot of f and pulling up.

$$f(\emptyset) = 0$$
$$f(\{1\}) = f(\{2\}) = 1$$
$$f(\{1,2\}) = 1$$

$$f^-(x_1, x_2) = \max(x_1, x_2)$$



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Claim

The convex closure exists for any set function.

Proof

- If $g_1, g_2 : [0,1]^n \to \mathbb{R}$ are convex under-estimators of f, then so is $\max\{g_1, g_2\}$
- Holds for infinite set of convex under-estimators
- Therefore $f^- = \max\{g : g$ is a convex underestimator of $f\}$ is the point-wise greatest convex underestimator of f.

Claim

The value of the convex closure at $x \in [0,1]^n$ is the solution of the following optimization problem:

$$
\begin{array}{ll}
\text{minimize} & \sum_{y \in \{0,1\}^n} \lambda_y f(y) \\
\text{subject to} & \sum_{y \in \{0,1\}^n} \lambda_y y = x \\
 & \sum_{y \in \{0,1\}^n} \lambda_y = 1 \\
 & \lambda_y \geq 0, \qquad \qquad \text{for } y \in \{0,1\}^n .
\end{array}
$$

Interpretation

- The minimum expected value of f over all distributions on $\{0,1\}^n$ with expectation x.
- Equivalently: the minimum expected value of f for a random set $S \subseteq X$ including each $i \in X$ with probability x_i.
- The upper bound on $f^-(x)$ implied by applying Jensen's inequality to every convex combination $\{0,1\}^n$.

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Implication

- f^- is a convex extension of f.
- $f^-(x)$ has no "integrality gap"
  - For every $x \in [0,1]^n$, there is a random integer vector $y \in \{0,1\}^n$ such that $\mathbf{E}_y\, f(y) = f^-(x)$.
  - Therefore, there is an integer vector y such that $f(y) \leq f^-(x)$.

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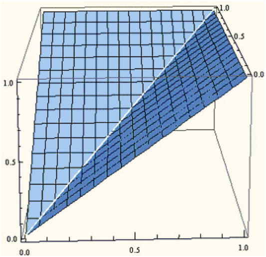
When $x_1 \leq x_2$

$$f^-(x_1, x_2) = x_1 f(\{1,2\}) + (x_2 - x_1) f(\{2\}) + (1 - x_2) f(\emptyset)$$



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$$\begin{array}{ll} \text{minimize} & \sum_{y \in \{0,1\}^n} \lambda_y f(y) \\ \text{subject to} & \sum_{y \in \{0,1\}^n} \lambda_y y = x \\ & \sum_{y \in \{0,1\}^n} \lambda_y = 1 \\ & \lambda_y \geq 0, \qquad\qquad \text{for } y \in \{0,1\}^n. \end{array}$$

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- Convex: The value of a minimization LP is convex in its right hand side constants (check)

Using the Convex Closure

Fact

The minimum of f^- is equal to the minimum of f, and moreover is attained at minimizers $y \in \{0,1\}^n$ of f.

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- Therefore $\min_{x \in [0,1]^n} f^-(x) \leq \min_{y \in \{0,1\}^n} f(y)$
- For every x, $f^-(x)$ is the expected value of $f(y)$, for a random variable $y \in \{0,1\}^n$ with expectation x.
- Therefore, $\min_{x \in [0,1]^n} f^-(x) \geq \min_{y \in \{0,1\}^n} f(y)$

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We reduced minimizing set function f to minimizing a convex function f^- over a convex set $[0,1]^n$. Are we done?

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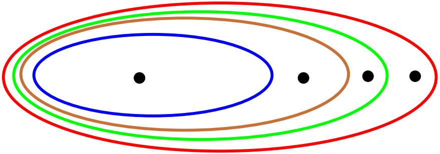
In general, it is hard to evaluate f^- efficiently, let alone its derivative. This is indispensible for convex optimization algorithms.

We will show that, when f is submodular, f^- is in fact equivalent to another extension which is easier to evaluate.

Chain Distributions

Chain Distribution

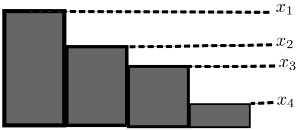
A chain distribution on the ground set X is a distribution over $S \subseteq X$ who's support forms a chain in the inclusion order.

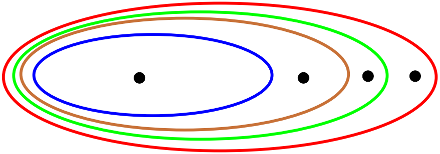


Chain Distributions

Chain Distribution with Given Marginals

Fix the ground set $X = \{1, \ldots, n\}$. The chain distribution with marginals $x \in [0,1]^n$ is the unique chain distribution $D^{\mathcal{L}}(x)$ satisfying $\mathbf{Pr}_{S \sim D^{\mathcal{L}}(x)}[i \in S] = x_i$ for all $i \in X$.





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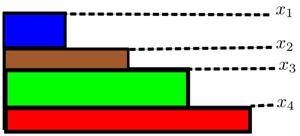

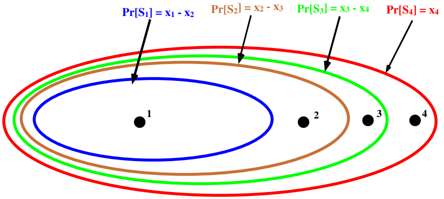

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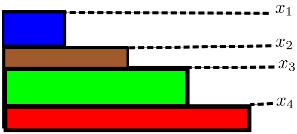

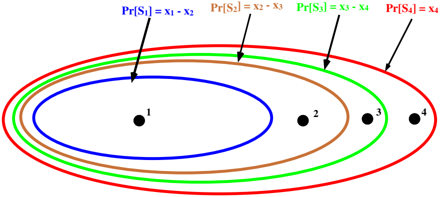

$D^{\mathcal{L}}(x)$ is the distribution given by the following process:

- Sort $x_1 \geq x_2 \ldots \geq x_n$
- Let $S_i = \{x_1, \ldots, x_i\}$
- Let $\mathbf{Pr}[S_i] = x_i - x_{i+1}$

The Lovasz Extension

Definition

The Lovasz extension of a set function f is defined as follows.

$$f^{\mathcal{L}}(x) = \mathop{\mathbf{E}}_{S \sim D^{\mathcal{L}}(x)} f(S)$$

i.e. the Lovasz extension at x is the expected value of a set drawn from the unique chain distribution with marginals x.

Observations

- $f^{\mathcal{L}}$ is an extension, since the chain distribution with marginals $y \in \{0,1\}^n$ is the point distribution at y.

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- Together, those imply: if $f^{\mathcal{L}}$ is convex, then $f^{\mathcal{L}} = f^{-}$.

Equivalence of the Convex Closure and Lovasz Extension

Theorem

If f is submodular, then $f^{\mathcal{L}} = f^{-}$.

Converse holds: if f is not submodular, then $f^{\mathcal{L}}$ is not convex.

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- Recall: $f^{-}(x)$ evaluates f on the “lowest” distribution with marginals x
- It turns out that, when f is submodular, this lowest distribution is the chain distribution $D^{\mathcal{L}}(x)$.

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Intuition

- Recall: $f^{-}(x)$ evaluates f on the "lowest" distribution with marginals x
- It turns out that, when f is submodular, this lowest distribution is the chain distribution $D^{\mathcal{L}}(x)$.
- Contingent on marginals x, submodularity implies that cost is minimized by "packing" as many elements together as possible
  - diminishing marginal returns
- This gives the chain distribution

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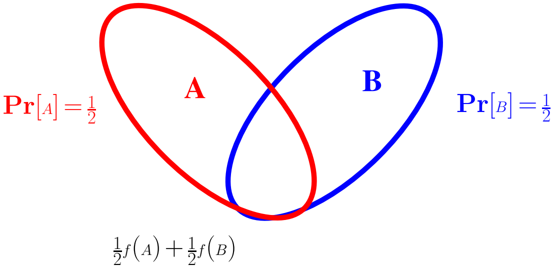

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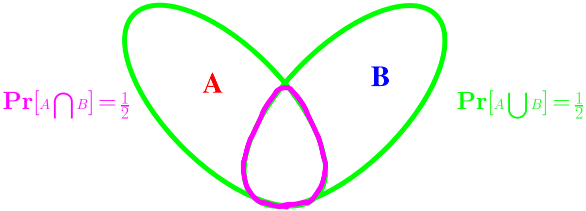
- Consider a distribution $\mathcal{D}$ on two "crossing" sets A and B, with probability 0.5 each.
- "uncrossing" implies that replacing them with $A \bigcap B$ and $A \bigcup B$, with probability 0.5 each, gives a chain distribution with lower expected value of f.



$$\frac{1}{2}f(A) + \frac{1}{2}f(B) \geq \frac{1}{2}f(A \bigcap B) + \frac{1}{2}f(A \bigcup B)$$

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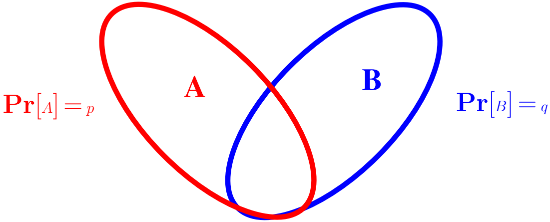
- Consider a distribution $\mathcal{D}$ on two "crossing" sets A and B, with probabilities $p \leq q$.



$pf(A) + qf(B)$

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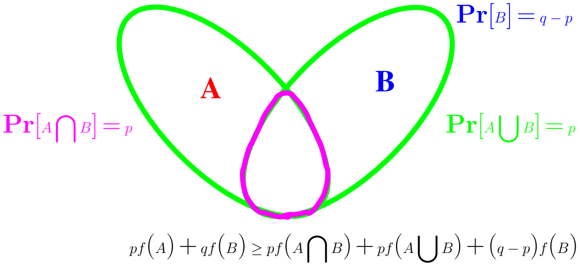
- Consider a distribution $\mathcal{D}$ on two "crossing" sets A and B, with probabilities $p \leq q$.
- Can "uncross" a probability mass of p of each, decreasing the expected value of f



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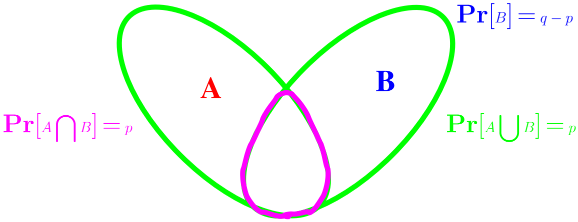
- Now a chain distribution



$$pf(A) + qf(B) \geq pf(A \bigcap B) + pf(A \bigcup B) + (q-p)f(B)$$

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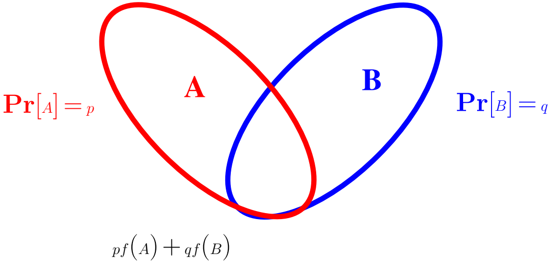

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- Consider a distribution $\mathcal{D}$ which includes two “crossing” sets A and B in its support
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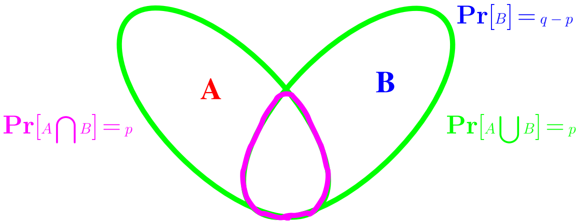

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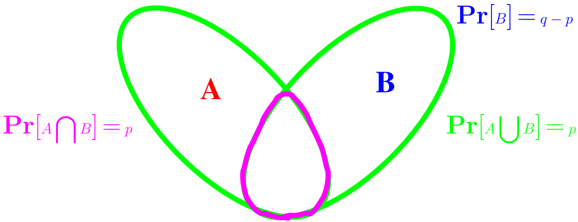
- Decreases number of crossing pairs of sets in the support.
  - Closer to being a chain distribution.



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Because $f^{\mathcal{L}} = f^{-}$, we know the following:

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Therefore, minimizing f reduces to the following convex optimization problem

Minimizing the Lovasz Extension

$$\begin{array}{ll} \text{minimize} & f^{\mathcal{L}}(x) \\ \text{subject to} & x \in [0,1]^n \end{array}$$

Recall: Solvability of Convex Optimization

Weak Solvability

An algorithm weakly solves our optimization problem if it takes in approximation parameter $\epsilon > 0$, runs in $\mathrm{poly}(n, \log \frac{1}{\epsilon})$ time, and returns $x \in [0,1]^n$ which is ϵ-optimal:

$$f^{\mathcal{L}}(x) \leq \min_{y \in [0,1]^n} f^{\mathcal{L}}(y) + \epsilon[\max_{y \in [0,1]^n} f^{\mathcal{L}}(y) - \min_{y \in [0,1]^n} f^{\mathcal{L}}(y)]$$

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Polynomial Solvability of CP

In order to weakly minimize $f^{\mathcal{L}}$, we need the following operations to run in $\text{poly}(n)$ time:

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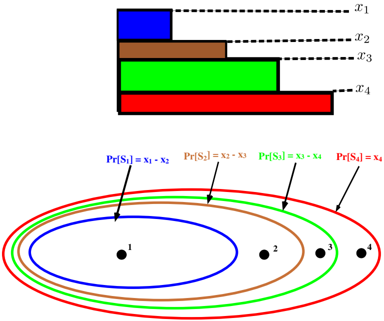
1 and 2 are trivial.

First order Oracle for $f^{\mathcal{L}}$



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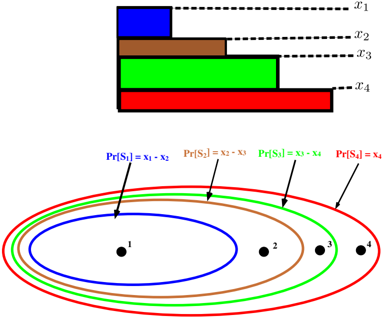

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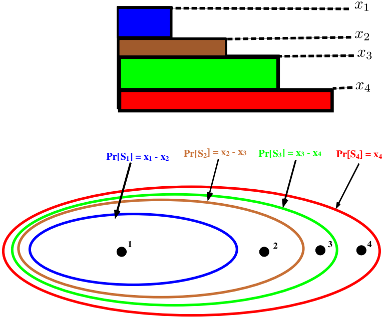

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We can get an ϵ-optimal solution x^* to the optimization problem in $\mathrm{poly}(n, \log \frac{1}{\epsilon})$ time.

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- We can identify this set by examining the chain distribution with marginals x^*

Outline

Recall: Optimizing Submodular Functions

	Maximization	Minimization
Unconstrained	NP-hard $\frac{1}{2}$ approximation	Polynomial time via convex opt
Constrained	Usually NP-hard $1 - 1/e$ (mono, matroid) $O(1)$ (“nice” constraints)	Usually NP-hard to apx. Few easy special cases

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Problem Definition

Given a non-decreasing and normalized submodular function $f : 2^X \to \mathbb{R}_+$ on a finite ground set X, and a matroid $M = (X, \mathcal{I})$

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- Non-decreasing: $f(S) \leq f(T)$ for $S \subseteq T$
- Normalized: $f(\emptyset) = 0$.
- We denote $n = |X|$

Representation

As before, we work in the value oracle and independence oracle models. Namely, we assume we have access to a subroutine evaluating $f(S)$, and a subroutine for checking whether $S \in \mathcal{I}$, each in constant time.

Examples

Maximum Coverage

X is the left hand side of a graph, and $f(S)$ is the total number of neighbors of S.

- Can think of $i \in X$ as a set, and $f(S)$ as the total “coverage” of S.

Goal is to cover as much of the RHS as possible with k LHS nodes.

Social Influence

- X is the family of nodes in a social network
- A meme, idea, or product is adopted at a set of nodes S
- $f(S)$ is the expected number of nodes in the network which end up adopting the idea.
- Goal is to obtain maximum influence subject to a constraint
  - Cardinality
  - Transversal
  - ...

Combinatorial Allocation

- G is a set of goods
- $f_i(B)$ is submodular utility of agent $i \in N$ for bundle $B \subseteq G$
- Allocation: A partition $(B_1, \ldots, B_n)$ of G among agents.
- Aggregate utility is $\sum_i f_i(B_i)$.

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- Allocation: A partition $(B_1, \ldots, B_n)$ of G among agents.
- Aggregate utility is $\sum_i f_i(B_i)$.
- Let $X = G \times N$ be the set of good/agent pairs
- Allocations correspond to subsets S of X in which at most one "copy" of each good is chosen
  - Partition matroid constraint
- $f(S) = \sum_{i \in N} f_i(\{j \in G : (j,i) \in X\})$
  - Submodular

Complexity

Theorem

Maximizing a submodular function subject to a matroid constraint is NP-hard, and NP-hard to approximate to within any better than a factor of $1 - 1/e$.

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An algorithm in the value oracle model which

- Runs in time $\mathrm{poly}(n)$
- Returns a feasible set $S^* \in \mathcal{I}$ satisfying $f(S^*) \geq (1 - 1/e) \max_{S \in \mathcal{I}} f(S)$.

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Holds for arbitrary matroid, but much simpler for uniform matroids.

Subject to a Cardinality Constraint

Problem Definition

Given a non-decreasing and normalized submodular function $f : 2^X \to \mathbb{R}_+$ on a finite ground set X with $|X| = n$, and an integer $k \leq n$

$$\begin{aligned} \text{maximize} \quad & f(S) \\ \text{subject to} \quad & |S| \leq k \end{aligned}$$

- k-uniform matroid constraint

The Greedy Algorithm

The following is the straightforward adaptation of the greedy algorithm for maximizing modular functions over a matroid.

The Greedy Algorithm

1. $S \leftarrow \emptyset$
2. While $|S| \leq k$
   - Choose $e \in X$ maximizing $f(S \bigcup \{e\})$
   - $S \leftarrow S \bigcup \{e\}$

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Theorem

The greedy algorithm is a $(1 - 1/e)$ *approximation algorithm for maximizing a monotone, normalized, and submodular function subject to a cardinality constraint.*

Contraction/Conditioning

Let $f : 2^X \to \mathbb{R}$ and $A \subseteq X$. Define $f_A(S) = f(A \bigcup S) - f(A)$.

Lemma

If f is monotone and submodular, then f_A is monotone, submodular, and normalized for any A.

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  - $f_A(S) = f(S \cup A) - f(A) \leq f(T \cup A) - f(A) = f_A(T)$.
- Submodular:

$$\begin{aligned} f_A(S) + f_A(T) &= f(S \cup A) - f(A) + f(T \cup A) - f(A) \\ &\geq f(S \cup T \cup A) - f(A) + f((S \cap T) \cup A) - f(A) \\ &= f_A(S \cup T) - f_A(S \cap T) \end{aligned}$$

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- Therefore, $\max_{j \in A} f(\{j\}) \geq \frac{1}{|A|} f(A)$

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- After k iterations, it has shrunk to $(1 - 1/k)^k \leq 1/e$ from its original value

$$OPT - f(S) \leq \frac{1}{e} OPT$$

$$(1 - 1/e) OPT \leq f(S)$$

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- Therefore, suboptimality decreases by factor of $1 - \frac{1}{k}$, as needed.

From Uniform to Arbitrary Matroid

Problem Definition

Given a non-decreasing and normalized submodular function $f : 2^X \to \mathbb{R}_+$ on a finite ground set X, and a matroid $M = (X, \mathcal{I})$

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  - It is, however, a $1/2$ approximation
- Nevertheless, a continuous greedy algorithm gives $1 - 1/e$
- Approach resembles that for minimization
  - Define a continous extension of f
  - Optimize continuous extension over matroid polytope
  - Extract an integer point

The Multilinear Extension

Multilinear Extension

Given a set function $f : \{0,1\}^n \to \mathbb{R}$, its multilinear extension $F : [0,1]^n \to \mathbb{R}$ evaluated at $x \in [0,1]^n$ gives the expected value of $f(S)$ for the random set S which includes each i independently with probability x_i.

$$F(x) = \sum_{S \subseteq X} f(S) \prod_{i \in S} x_i \prod_{i \neq S} (1 - x_i)$$

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- For each point x, evaluates f on the independent distribution $D(x)$
- Clearly an extension of f
- Not concave (or convex) in general
  - Recall f with $f(\emptyset) = 0$ and $f(\{1\}) = f(\{2\}) = f(\{1,2\}) = 1$
  - $F(x) = 1 - (1 - x_1)(1 - x_2)$

Easy Properties of the Multilinear Extension

Normalized

When f is normalized, $F(0) = 0$

Follows from the fact that F is an extension of f

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Nondecreasing

When f is monotone non-decreasing, $F(x) \leq F(y)$ whenever $x \preceq y$ component-wise.

Increasing the probability of selecting each element increases the expected value.

Up-concavity

Even though F is not concave, it is concave in "upwards" directions.

Up-concavity

Assume f is submodular. For every $\vec{a} \in [0,1]^n$ and $\vec{d} \in [0,1]^n$ satisfying $d \succeq 0$, the function $F(\vec{a} + \vec{d}\, t)$ is a concave function of $t \in \mathbb{R}$.

- This follows almost directly from diminishing marginal returns interpretation of submodularity.
- Proof sketch:
  - Up concave $\equiv$ mixed derivatives $\frac{\partial^2 F}{\partial x_i \partial x_j}$ negative everywhere
  - Negative mixed derivatives follow from diminishing marginal returns

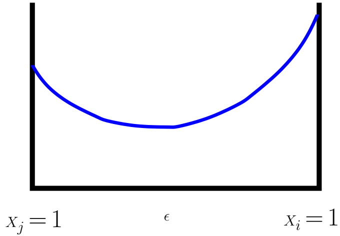
Cross-convexity

Nevertheless, F is convex in "cross" directions.

Cross-convexity

Assume f is submodular. For every $a \in [0,1]^n$ and $\vec{d} = e_i - e_j$ for some $i, j \in X$, the function $F(\vec{a} + \vec{d}\, t)$ is a convex function of $t \in \mathbb{R}$.

- i.e. trading off one item's probability for anothers gives a convex curve
- Follows from submodularity: as we "remove" j, the marginal benefit of "adding" i increases



Algorithm Outline

Step A: Continuous Greedy Algorithm

Computes a $1 - 1/e$ approximation to the following continuous (non-convex) optimization problem.

$$\begin{aligned} \text{maximize} \quad & F(x) \\ \text{subject to} \quad & x \in \mathcal{P}(\mathcal{M}) \end{aligned}$$

- i.e. Computes x^* s.t. $F(x^*) \geq (1 - 1/e) \max \{F(x) : x \in \mathcal{P}(\mathcal{M})\}$

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- Would we be done?

No! $D(x^*)$ may be mostly supported on infeasible sets (i.e. not independent in matroid $\mathcal{M}$).

Algorithm Outline

Step B: Pipage Rounding

"Rounds" x^* to some vertex y^* of the matroid polytope (i.e. an independent set) satisfying

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- A-priori, not obvious that such a y^* exists

- The following "continuous" descent algorithm works for an arbitrary nondecreasing and up-concave function F, and solvable downwards-closed polytope $\mathcal{P} \subseteq \mathbb{R}_+^n$.
- Continuously moves a particle inside the matroid polytope, starting at 0, for a total of 1 time unit.
  - Position at time t given by $x(t)$.
- Discretized to time steps of ϵ, which we will assume to be arbitrarily small for convenience of analysis, but may be taken to be $1/\operatorname{poly}(n)$ in the actual implementation.

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Continuous Greedy Algorithm (F,$\mathcal{P}$, ϵ)

1. $x(0) \leftarrow \vec{0}$
2. For $t \in [0, \epsilon, 2\epsilon, \ldots, 1-\epsilon]$
   - $x(t+\epsilon) \leftarrow x(t) + \epsilon \operatorname{argmax}_{y \in \mathcal{P}} \{\triangledown F(x(t)) \cdot y\}$
3. Return $x(1)$

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- I.e. When the particle is at x, it moves in direction y maximizing the linear function $\triangledown F(x) \cdot y$ over $y \in \mathcal{P}$
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- Observe: Algorithm forms a convex combination of $\frac{1}{\epsilon}$ vertices of the polytope $\mathcal{P}$, each with weight ϵ.
  - $x(1) \in \mathcal{P}$.

Theorem

Let F be nondecreasing and up-concave, and $\mathcal{P}$ be a downwards closed polytope. In the limit as $\epsilon \to 0$, the continuous greedy algorithm outputs a $1 - 1/e$ approximation to maximizing $F(x)$ over $\mathcal{P}$.

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$$\geq OPT - F(x(t))$$

Theorem

Let F be nondecreasing and up-concave, and $\mathcal{P}$ be a downwards closed polytope. In the limit as $\epsilon \to 0$, the continuous greedy algorithm outputs a $1 - 1/e$ approximation to maximizing $F(x)$ over $\mathcal{P}$.

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- Denote $y(t) = \text{argmax}_{y \in \mathcal{P}} \bigtriangledown F(x(t)) \cdot y$
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Proof Sketch

- $v(t) = F(x(t))$ satisfies $\frac{dv}{dt} \geq OPT - v$.
- Differential equation $\frac{dv}{dt} = OPT - v$ with boundary condition $v(0) = 0$ has a unique solution

$$v(t) = OPT(1 - e^{-t})$$

- $v(1) \geq OPT(1 - 1/e)$

Implementation Details

Continuous Greedy Algorithm (F,$\mathcal{P}$, ϵ)

1. $x(0) \leftarrow \vec{0}$
2. For $t \in [0, \epsilon, 2\epsilon, \ldots, 1-\epsilon]$
   - $x(t+\epsilon) \leftarrow x(t) + \epsilon \operatorname{argmax}_{y \in \mathcal{P}} \{\bigtriangledown F(x(t)) \cdot y\}$
3. Return $x(1)$

- $\bigtriangledown F(x)$ is not readily available, but can be estimated "accurately enough" using $\text{poly}(n)$ random samples from $D(x)$, w.h.p.
- Step 2 can be implemented because $\mathcal{P}$ is solvable
- Discretization: Taking $\epsilon = 1/O(n^2)$ is "fine enough"
- Both the above introduce error into the approximation guarantee, yielding $1 - 1/e - 1/O(n)$ w.h.p
- This can be shaved off to $1 - 1/e$ with some additional "tricks".

- The following algorithm takes x in matroid base polytope $\mathcal{P}_{base}(\mathcal{M})$, and non-decreasing cross-convex function F, and outputs integral y with $F(y) \geq F(x)$

PipageRounding ($\mathcal{M}$,x, F)

While x contains a fractional entry

1. Let T be a minimum-size tight set containing some fractional entry
   - i.e. $x(T) = rank_{\mathcal{M}}(T)$, and some $i \in T$ satisfies $x_i \in (0,1)$.
2. Let $j \in T$ be such that $j \neq i$ and x_j is fractional.
3. Let $x(\mu) = x + \mu(e_i - e_j)$, and maximize $F(x(\mu))$ subject to $x(\mu) \in \mathcal{P}(\mathcal{M})$.
4. $x \leftarrow x(\mu)$.

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Theorem

On input $x \in \mathcal{P}_{base}(\mathcal{M})$, Pipage rounding terminates in $O(n^2)$ iterations, and outputs a matroid vertex y with $f(y) = F(y) \geq F(x)$

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Step 1

- T is the minimum tight set including i, because tight sets with respect to $\mathcal{P}(\mathcal{M})$ form a lattice
- Proof:
  - Tight sets in x are the minimizers of the set function $rank_{\mathcal{M}}(S) - x(S)$
  - This set function is submodular.
  - Minimizers of a submodular function form a lattice (implied by submodular inequality).

PipageRounding ($\mathcal{M}$,x, F)

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Step 2

- Since rank is integer valued, any tight set containing fractional variable should have another.

PipageRounding ($\mathcal{M}$,x, F)

While x contains a fractional entry

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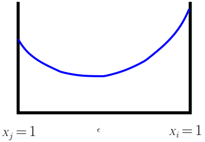
4. $x \leftarrow x(\mu)$.

Step 3

- Either the number of fractional variables decreases, or a smaller tight set containing x_i or x_j is created.
- This leads to termination after $O(n^2)$ iterations
- By cross convexity, objective increases



To summarize

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Let F be nondecreasing and up-concave, and $\mathcal{P}$ be a downwards closed polytope. In the limit as $\epsilon \to 0$, the continuous greedy algorithm outputs a $1 - 1/e$ approximation to maximizing $F(x)$ over $\mathcal{P}$.

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Together, these imply a $1 - 1/e$ approximation algorithm for monotone submodular maximization subject to a matroid constraint