

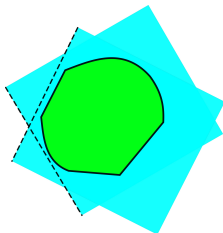
CS675: Convex and Combinatorial Optimization
Fall 2016
Duality of Convex Sets and Functions

Instructor: Shaddin Dughmi

Outline

- 1 Convexity and Duality
- 2 Duality of Convex Sets
- 3 Duality of Convex Functions

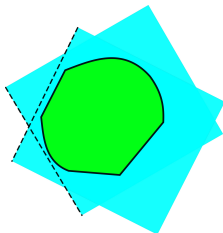
Duality Correspondences



There are two equivalent ways to represent a convex set

- The family of points in the set (standard representation)
- The set of halfspaces containing the set (“dual” representation)

Duality Correspondances

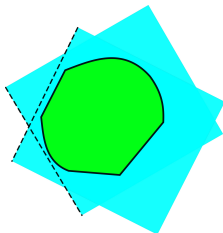


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This equivalence between the two representations gives rise to a variety of “duality” relationships among convex sets, cones, and functions.

Duality Correspondances



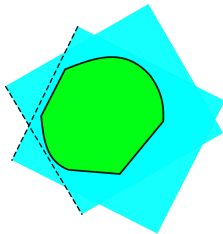
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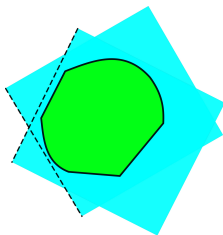
Definition

“**Duality**” is a woefully overloaded mathematical term for a relation that groups elements of a set into “dual” pairs.



Theorem

A closed convex set S is the intersection of all closed halfspaces \mathcal{H} containing it.

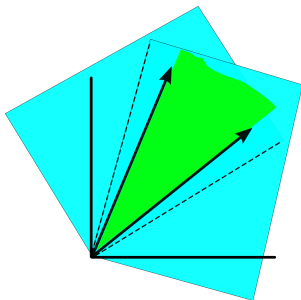


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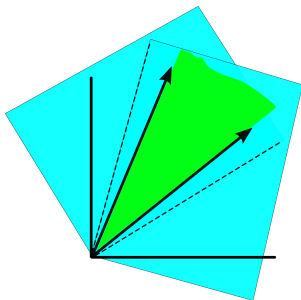
Proof

- Clearly, $S \subseteq \bigcap_{H \in \mathcal{H}} H$
- To prove equality, consider $x \notin S$
- By the separating hyperplane theorem, there is a hyperplane separating S from x
- Therefore there is $H \in \mathcal{H}$ with $x \notin H$, hence $x \notin \bigcap_{H \in \mathcal{H}} H$



Theorem

A closed convex cone K is the intersection of all closed homogeneous halfspaces \mathcal{H} containing it.

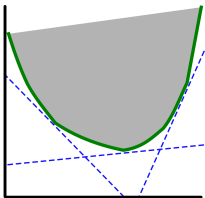


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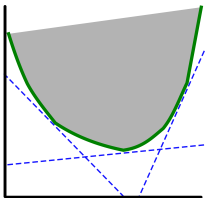
Proof

- For every non-homogeneous halfspace $a^\top x \leq b$ containing K , the smaller homogeneous halfspace $a^\top x \leq 0$ contains K as well.
- Therefore, can discard non-homogeneous halfspaces when taking the intersection



Theorem

A convex function is the point-wise supremum of all affine functions under-estimating it everywhere.



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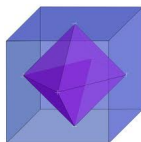
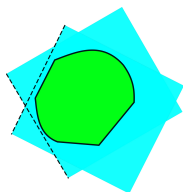
Proof

- $\text{epi } f$ is convex
- Therefore $\text{epi } f$ is the intersection of a family of halfspaces of the form $a^\top x - t \leq b$, for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- Each such halfspace constrains $(x, t) \in \text{epi } f$ to $a^\top x - b \leq t$
- $f(x)$ is the lowest t s.t. $(x, t) \in \text{epi } f$
- Therefore, $f(x)$ is the point-wise maximum of $a^\top x - b$ over all halfspaces

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Polar Duality of Convex Sets



One way of representing the all halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The **polar** of S is defined as follows:

$$S^\circ = \{y : y^\top x \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $a^\top x \leq b$ with $b \neq 0$ can be written as a “normalized” inequality $y^\top x \leq 1$, by dividing by b .
- S° can be thought of as the normalized representations of halfspaces containing S .

$$S^\circ = \{y : y^\top x \leq 1 \text{ for all } x \in S\}$$

Properties of the Polar

- 1 $S^{\circ\circ} = S$
- 2 S° is a closed convex set containing the origin
- 3 When 0 is in the interior of S , then S° is bounded.

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Properties of the Polar

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-
- 2 Follows from representation as intersection of halfspaces
 - 3 S contains an ϵ -ball centered at the origin, so $\|y\| \leq 1/\epsilon$ for all $y \in S^\circ$.

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Properties of the Polar

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- 1
 - Easy to see that $S \subseteq S^{\circ\circ}$
 - Take $x_o \notin S$, by SSHT and $0 \in S$, there is a halfspace $z^\top x \leq 1$ containing S but not x_o (i.e. $z^\top x_o > 1$)
 - $z \in S^\circ$, therefore $x \notin S^{\circ\circ}$

$$S^\circ = \{y : y^T x \leq 1 \text{ for all } x \in S\}$$

Properties of the Polar

- 1 $S^{\circ\circ} = S$
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Note

When S is non-convex, $S^\circ = (\text{convexhull}(S))^\circ$, and $S^{\circ\circ} = \text{convexhull}(S)$.

Examples



The unit sphere for different metrics: $\|x\|_{l_p} = 1$ in \mathbb{R}^2 .

Norm Balls

- The polar of the Euclidean unit ball is itself (we say it is **self-dual**)
- The polar of the 1-norm ball is the ∞ -norm ball
- More generally, the p -norm ball is dual to the q -norm ball, where

$$\frac{1}{p} + \frac{1}{q} = 1$$

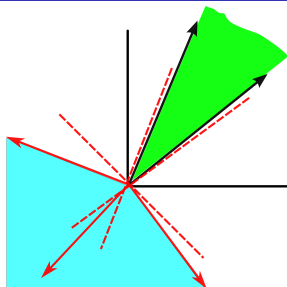


Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A .

- Facets of P correspond to vertices of P° .
- Dually, vertices of P correspond to facets of P° .

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

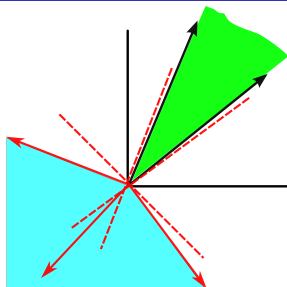
The polar of a closed convex cone K is given by

$$K^\circ = \{y : y^\top x \leq 0 \text{ for all } x \in K\}$$

Note

- If halfspace $y^\top x \leq b$ contains K , then so does smaller $y^\top x \leq 0$.
- K° represents all homogeneous halfspaces containing K .

Polar Duality of Convex Cones



Polar duality takes a simplified form when applied to cones

Polar

The polar of a closed convex cone K is given by

$$K^\circ = \{y : y^\top x \leq 0 \text{ for all } x \in K\}$$

Dual Cone

By convention, $K^* = -K^\circ$ is referred to as the **dual cone** of K .

$$K^* = \{y : y^\top x \geq 0 \text{ for all } x \in K\}$$

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Properties of the Polar Cone

- 1 $K^{\circ\circ} = K$
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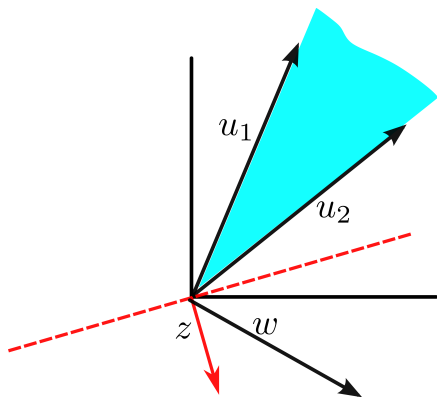
- 1 Same as before
- 2 Intersection of homogeneous halfspaces

Examples

- The polar of a subspace is its orthogonal complement
- The polar cone of the nonnegative orthant is the nonpositive orthant
 - Self-dual
- The polar of a polyhedral cone $Ax \preceq 0$ is the conic hull of the rows of A
- The polar of the cone of positive semi-definite matrices is the cone of negative semi-definite matrices
 - Self-dual

Recall: Farkas' Lemma

Let K be a **closed convex cone** and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^\top x \leq 0$ for all $x \in K$, and $z^\top w > 0$.

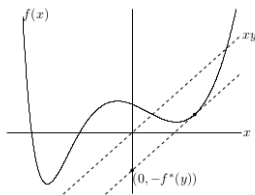


Equivalently: there is $z \in K^\circ$ with $z^\top w > 0$.

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Conjugation of Convex Functions



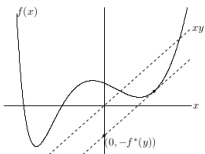
Conjugate

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a convex function. The **conjugate** of f is

$$f^*(y) = \sup_x (y^\top x - f(x))$$

Note

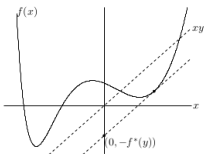
- $f^*(y)$ is the minimal value of β such that the affine function $y^\top x - \beta$ underestimates $f(x)$ everywhere.
- Equivalently, the distance we need to lower the hyperplane $y^\top x - t = 0$ in order to get a supporting hyperplane to $\text{epi } f$.
- $y^\top x - t = f^*(y)$ are the supporting hyperplanes of $\text{epi } f$



$$f^*(y) = \sup_x (y^\top x - f(x))$$

Properties of the Conjugate

- 1 $f^{**} = f$ when f is convex
- 2 f^* is a convex function
- 3 $xy \leq f(x) + f^*(y)$ for all $x, y \in \mathbb{R}^n$ (Fenchel's Inequality)

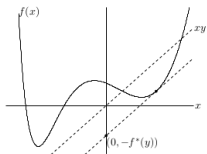


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- 2 Supremum of affine functions of y
- 3 By definition of $f^*(y)$

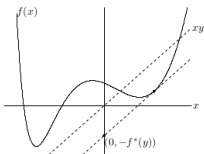


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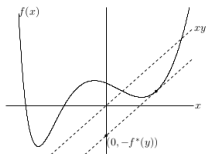
- 1 • $f^{**}(x) = \max_y y^\top x - f^*(y)$ when f is convex



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Properties of the Conjugate

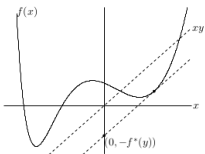
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- $f^{**}(x) = \max_y y^\top x - f^*(y)$ when f is convex
 - For fixed y , $f^*(y)$ is minimal β such that $y^\top x - \beta$ underestimates f .



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 - $f^{**}(x) = \max_y y^\top x - f^*(y)$ when f is convex
 - For fixed y , $f^*(y)$ is minimal β such that $y^\top x - \beta$ underestimates f .
 - Therefore $f^{**}(x)$ is the maximum, over all y , of affine underestimates $y^\top x - \beta$ of f



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 - 3 Therefore $f^{**}(x)$ is the maximum, over all y , of affine underestimates $y^\top x - \beta$ of f
 - 4 By our characterization early in this lecture, this is equal to f .

Examples

- Affine function: $f(x) = ax + b$. Conjugate has $f^*(a) = b$, and ∞ elsewhere
- $f(x) = x^2/2$ is self-conjugate
- Exponential: $f(x) = e^x$. Conjugate has $f^*(y) = y \log y - y$ for $y \in \mathbb{R}_+$, and ∞ elsewhere.
- Quadratic: $f(x) = \frac{1}{2}x^\top Qx$ with $Q \succeq 0$. Self conjugate.
- Log-sum-exp: $f(x) = \log(\sum_i e^{x_i})$. Conjugate has $f^*(y) = \sum_i y_i \log y_i$ for $y \succeq 0$ and $1^\top y = 1$, ∞ otherwise.