# CS675: Convex and Combinatorial Optimization
# Fall 2019
# The Ellipsoid Algorithm

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# History and Basics

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- In 1979, Kachiyan shows that it yields a polynomial time algorithm for LP
- Inefficient in practice, has been surpassed by newer interior point methods.
- Theoretically most powerful, with deep consequences for complexity and optimization
  - Polynomial-time algorithm for linear programming
  - Polynomial-time algorithm for approximate convex optimization under mild conditions
  - Equivalence of separation and optimization

# Outline

## Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing $K$
- A rational number $R > 0$ satisfying $\mathbf{vol}(E) \leq R$.
- A rational number $r > 0$ such that if $K$ is nonempty, then $\mathbf{vol}(K) \geq r$.

Find a point $x \in K$ or declare that $K$ is empty.

- Note: convex optimization reduces to checking feasibility by binary search
  - We will explain later how the parameters $r, R, E$ figure into this
- Description of $K$: any description that admits an efficient implementation of a separation oracle

## Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting $x$ from $K$.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal y < h^\intercal x$ for all $y \in K$.
- Equivalently, $K$ is contained in the open halfspace

  $$H(h, x) = \{y : h^\intercal y < h^\intercal x\}$$

  with $x$ at its boundary.

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- Convex set given by a family of convex inequalities $f_i(y) \leq 0$: Let $h = \bigtriangledown f_i(x)$ for some violated constraint.

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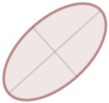
- The positive semi-definite cone $S_n^+$: Let $H = -vv^\intercal$, where $v$ is an eigenvector corresponding to a negative eigenvalue.

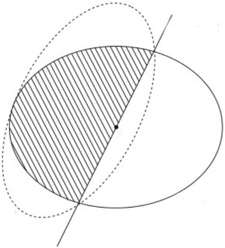
# Recall:Ellipsoids



- Unit ball $B(0,1) = \{x \in \mathbb{R}^n : x^\intercal x \leq 1\}$.
- Ellipsoid: the result of applying some affine transformation $x \to Lx + c$ to $B(0,1)$.
  - Gives the set $\left\{x : (x-c)^\intercal L^{-\intercal} L^{-1} (x-c) \leq 1\right\}$
- It is conventional to take $Q = LL^\intercal$, which is symmetric PSD, and write the ellipsoid as

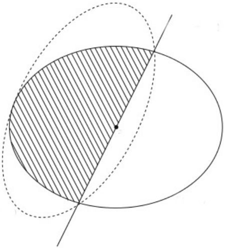
$$E(c,Q) = \left\{x : (x-c)^\intercal Q^{-1} (x-c) \leq 1\right\}$$

- Can calculate volume easily: $\mathbf{vol}(E(c,Q)) = \sqrt{\det Q}\,\mathbf{vol}(B(0,1))$
- Note: since $Q \succeq 0$ has a PSD square root, can take $L$ to be PSD as well



## Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output $c$.
   - Otherwise, we get a separating hyperplane $h$ such that $K$ is contained in the half-ellipsoid $E \bigcap \{y : h^\intercal y \leq h^\intercal c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



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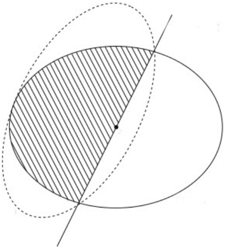

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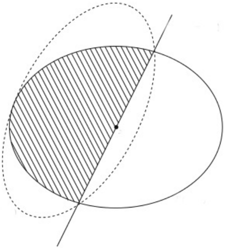

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# Outline

# Correctness

**Theorem**

If the ellipsoid algorithm terminates, then it either outputs $x \in K$ or correctly declares that $K$ is empty.

- Algorithm outputs some $x$ only if it was certified in $K$ by the separation oracle
- $E \supseteq K$ is maintained throughout the algorithm, by definition
- We are promised $\mathbf{vol}(K) < r$ only if $K = \emptyset$
- If algorithm outputs empty then $r > \mathbf{vol}(E) \geq \mathbf{vol}(K)$, and $K = \emptyset$.

# Runtime

To prove that the ellipsoid method runs in polynomial time, we need the following lemma

## Lemma

- The minimum volume ellipsoid $E'$ containing half of another ellipsoid $E$ can be computed in $\mathrm{poly}(n)$ operations.
- Moreover, the volume of $E'$ is smaller than the volume of $E$ by a factor of at least $e^{\frac{1}{2(n+1)}} \approx 1 + \frac{1}{2(n+1)}$.

This is tricky to show, but there is an explicit, easy-to-compute, closed form for the matrix and center of $E'$ in terms of the matrix and center of $E$. (See notes for proof)

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The ellipsoid method terminates after $2(n+1)\ln\frac{R}{r}$ iterations. Moreover, each iteration can be implemented using $\mathrm{poly}(n)$ operations and a single call to the separation oracle.

- Volume of working ellipsoid starts off at $R$
- Decreases by a factor of $e^{\frac{1}{2(n+1)}}$ each iteration.
- After $2(n+1)\ln\frac{R}{r}$ interations, volume is less than $r$, triggering termination.

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- After $2(n+1)\ln\frac{R}{r}$ interations, volume is less than $r$, triggering termination.
- In each iteration, we
  - Call the separation oracle
  - Compute the minimum volume ellipsoid ($\mathrm{poly}(n)$ time by lemma)
  - Compute the volume of $E'$ (reduces to determinant computation).