CS675: Convex and Combinatorial Optimization
Fall 2019
Consequences of the Ellipsoid Algorithm

Instructor: Shaddin Dughmi

Outline

Recall: Feasibility Problem

The ellipsoid method solves the following problem.

Convex Feasibility Problem

Given as input the following

- A description of a compact convex set $K \subseteq \mathbb{R}^n$
- An ellipsoid $E(c, Q)$ (typically a ball) containing K
- A rational number $R > 0$ satisfying $\mathbf{vol}(E) \leq R$.
- A rational number $r > 0$ such that if K is nonempty, then $\mathbf{vol}(K) \geq r$.

Find a point $x \in K$ or declare that K is empty.

- Equivalent variant: drop the requirement on volume $\mathbf{vol}(K)$, and either find a point $x \in K$ or an ellipsoid $E \supseteq K$ with $\mathbf{vol}(E) < r$.

All the ellipsoid method needed was the following subroutine

Separation oracle

An algorithm that takes as input $x \in \mathbb{R}^n$, and either certifies $x \in K$ or outputs a hyperplane separting x from K.

- i.e. a vector $h \in \mathbb{R}^n$ with $h^\intercal x \geq h^\intercal y$ for all $y \in K$.
- Equivalently, K is contained in the open halfspace

$$H(h, x) = \{y : h^\intercal y < h^\intercal x\}$$

with x at its boundary.

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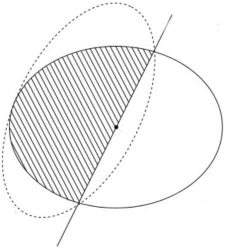
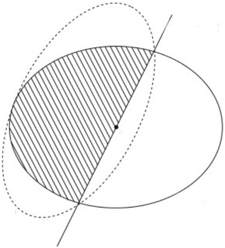
- The positive semi-definite cone S_n^+: Let H be $-vv^\intercal$ for an eigenvector v with a negative eigenvalue.



Ellipsoid Method

1. Start with initial ellipsoid $E = E(c, Q) \supseteq K$
2. Using the separation oracle, check if the center $c \in K$.
   - If so, terminate and output c.
   - Otherwise, we get a separating hyperplane h such that K is contained in the half-ellipsoid $E \bigcap \{y : h^{\intercal} y \leq h^{\intercal} c\}$
3. Let $E' = E(c', Q')$ be the minimum volume ellipsoid containing the half ellipsoid above.
4. If $\mathbf{vol}(E') \geq r$ then set $E = E'$ and repeat (step 2), otherwise stop and return "empty".



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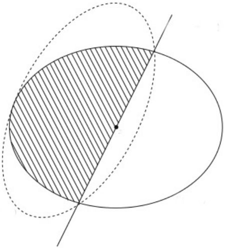

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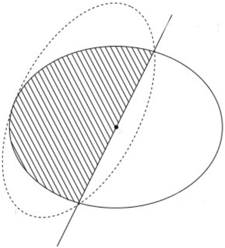

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Properties

Using T to denote the runtime of the separation oracle

Theorem

The ellipsoid algorithm terminates in time polynomial n, $\ln \frac{R}{r}$, and T, and either outputes $x \in K$ or correctly declares that K is empty.

We proved most of this (modulo the ellipsoid updating Lemma which we cited and briefly discussed).

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Note

For runtime polynomial in input size we need

- T polynomial in input size
- $\frac{R}{r}$ exponential in input size

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Recall: Convex Optimization Problem

A problem of minimizing a convex function (or maximizing a concave function) over a convex set.

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

Where $\mathcal{X} \subseteq \mathbb{R}^n$ is convex and closed, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex

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- Recall: A problem Π is a family of instances $I = (f, \mathcal{X})$
- When represented explicitly, often given in standard form

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & \text{for } i \in \mathcal{C}_1. \\ & a_i^\intercal x = b_i, & \text{for } i \in \mathcal{C}_2. \end{array}$$

- The functions f,$\{g_i\}_i$ are given in some parametric form allowing evaluation of each function and its derivatives.

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- We will abstract away details of how instances of a problem are represented, but denote the length of the description by $\langle I \rangle$
- Require polynomial time (in $\langle I \rangle$ and n) implementation of separation oracle, and other subroutines.

Solvability of Convex Optimization

There are many subtly different "solvability statements". This one is the most useful, yet simple to describe, IMO.

Requirements

We say an algorithm weakly solves a convex optimization problem in polynomial time if it:

- Takes an approximation parameter $\epsilon > 0$
- Terminates in time $\mathrm{poly}(\langle I \rangle, n, \log(\frac{1}{\epsilon}))$
- Returns an ϵ-optimal $x \in \mathcal{X}$:

$$f(x) \leq \min_{y \in \mathcal{X}} f(y) + \epsilon[\max_{y \in \mathcal{X}} f(y) - \min_{y \in \mathcal{X}} f(y)]$$

Solvability of Convex Optimization

Theorem (Polynomial Solvability of CP)

Consider a family Π of convex optimization problems $I = (f, \mathcal{X})$ admitting the following operations in polynomial time (in $\langle I \rangle$ and n):

- *A separation oracle for the feasible set $\mathcal{X} \subseteq \mathbb{R}^n$*
- *A first order oracle for f: evaluates $f(x)$ and $\triangledown f(x)$.*
- *An algorithm which computes a starting ellipsoid $E \supseteq \mathcal{X}$ with $\frac{\mathbf{vol}(E)}{\mathbf{vol}(\mathcal{X})} = O(\exp(\langle I \rangle, n))$.*

Then there is a polynomial time algorithm which weakly solves Π.

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Let's now prove this, by reducing to the ellipsoid method

Proof (Simplified)

Simplifying Assumption

Assume we are given $\min_{y\in\mathcal{X}} f(y)$ and $\max_{y\in\mathcal{X}} f(y)$. Without loss of generality assume they are $[0, 1]$.

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We can feed this into the Ellipsoid method!

Needed Ingredients

1. Separation oracle for new feasible set K:
2. Ellipsoid E containing K:
3. Guarantee that $\frac{\mathbf{vol}(E)}{\mathbf{vol}(K)} \leq \exp(n, \langle I \rangle, \log \frac{1}{\epsilon})$:

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$\mathbf{vol}(K) \geq \epsilon^n \, \mathbf{vol}(\mathcal{X})$.

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- Let $y = \epsilon x$ for $x \in \mathcal{X}$, and invoke Jensen's inequality

$$f(y) = f(\epsilon x + (1-\epsilon)0) \leq \epsilon f(x) + (1-\epsilon) f(0) \leq \epsilon$$

Proof (General)

- Denote $L = \min_{y \in \mathcal{X}} f(y)$ and $H = \max_{y \in \mathcal{X}} f(y)$
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- We don't need to know anything about T!

Key Observation

We don't really need to know T, H, or L to simulate the same execution of the ellipsoid method on K!!

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- Require separation oracle for K to use $\bigtriangledown f$ only as a last resort
  - This is allowed.
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- Run this simulation until enough iterations have passed, and take the best feasible point encountered. This must be in K.

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- In the implicit case, we require polynomial time in the bit complexity of individual entries of A, b, c.

Theorem (Polynomial Solvability of Explicit LP)

There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.

Proof Sketch (Informal)

Using result for weakly solving convex programs, we need 4 things:

- A separation oracle for $Ax \preceq b$: trivial when explicitly represented
- A first order oracle for $c^{\intercal}x$: also trivial
- A bounding ellipsoid of volume at most an exponential times the volume of the feasible polyhedron: tricky
- A way of "rounding" an ϵ-optimal solution to an optimal vertex solution: tricky

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Solution to both issues involves tedious accounting of numerical issues

Ellipsoid and Volume Bound (Informal)

Key to tackling both difficulties is the following observation:

Lemma

Let v be vertex of the polyhedron $Ax \preceq b$. It is the case that v has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\mathrm{poly}(\langle A \rangle, \langle b \rangle))$.

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

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Let v be vertex of the polyhedron $Ax \preceq b$. It is the case that v has polynomial bit complexity, i.e. $\langle v \rangle \leq M$, where $M = O(\text{poly}(\langle A \rangle, \langle b \rangle))$.

Specifically, the solution of a system of linear equations has bit complexity polynomially related to that of the equations.

- Bounding ellipsoid: all vertices contained in the box $-2^M \leq x \leq 2^M$, which in turn is contained in an ellipsoid of volume exponential in M and n.

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- To guarantee volume lowerbound, need to instead solve a "relaxed problem". Specifically, relaxing to $Ax \preceq b + \epsilon$, for sufficiently small ϵ with $\langle \epsilon \rangle = \mathrm{poly}(M)$. Gives volume exponentially small in M, but no smaller. Still close enough to original polyhedron so solution to relaxed problem can be "rounded" to solution of the original problem.

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- Rounding to a vertex: If a point y is ϵ-optimal for the ϵ-relaxed problem, for sufficiently small ϵ chosen carefully to polynomial in description of input, then rounding to the nearest x with M bits recovers the vertex.

Theorem (Polynomial Solvability of Implicit LP)

Consider a family Π *of linear programming problems* $I = (A, b, c)$ *admitting the following operations in polynomial time (in* $\langle I \rangle$ *and* n*):*

- *A separation oracle for the polyhedron* $Ax \preceq b$
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Moreover, assume that every $\langle a_{ij} \rangle$, $\langle b_i \rangle$, $\langle c_j \rangle$ *are at most* $\mathrm{poly}(\langle I \rangle, n)$*. Then there is a polynomial time algorithm for* Π *(both primal and dual*).*

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  - Turns out this is still OK, but takes a lot of work (see references).

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For the dual, we need equivalence of separation and optimization. Also, we necessarily get a solution to a normalized version of the dual. (HW)

Outline

Separation and Optimization

- One interpretation of the previous theorem is that optimization of linear functions over a polytope of polynomial bit complexity reduces to implementing a separation oracle
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Lets formalize the two questions, parametrized by a polytope P.

Linear Optimization Problem

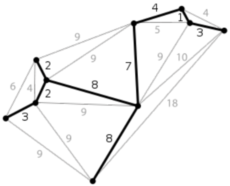
- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $\operatorname{argmax}_{x \in P} c^\intercal x$.

Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P$, or else find $h \in \mathbb{R}^n$ s.t. $h^\intercal x < h^\intercal y$ for all $x \in P$.

Recall: Minimum Cost Spanning Tree

Given a connected undirected graph $G = (V, E)$, and costs c_e on edges e, find a minimum cost spanning tree of G.



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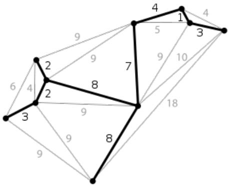

Spanning Tree Polytope

$$\begin{aligned} \sum_{e \subseteq X} x_e &\leq |X| - 1, && \text{for } X \subset V. \\ \sum_{e \in E} x_e &= n - 1 \\ x_e &\geq 0, && \text{for } e \in E. \end{aligned}$$

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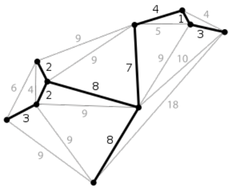

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- Optimization: Find the minimum/maximum weight spanning tree
- Separation: Find $X \subset V$ with $\sum_{e \subseteq X} x_e > |X| - 1$, if one exists
  - i.e. When edge weights are x, find a "dense" subgraph

Theorem (Equivalence of Separation and Optimization for Polytopes)

Consider a family $\mathcal{P}$ of polytopes $P = \{x : Ax \leq b\}$ described implicitly using $\langle P \rangle$ bits, and satisfying $\langle a_{ij} \rangle, \langle b_i \rangle \leq \mathrm{poly}(\langle P \rangle, n)$. Then the separation problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle y \rangle)$ time for $P \in \mathcal{P}$ if and only if the linear optimization problem is solvable in $\mathrm{poly}(\langle P \rangle, n, \langle c \rangle)$ time.

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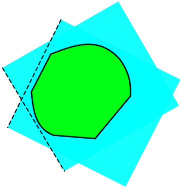
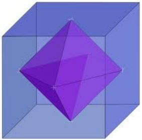
- E.g. Spanning tree polytopes, represented by graphs, are solvable.
- We already saw the the proof of the forward direction, via Ellipsoid method
  - Separation $\Rightarrow$ optimization
- For the other direction, we need polars

Recall: Polar Duality of Convex Sets





One way of representing the all halfspaces containing a convex set.

Polar

Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The polar of S is defined as follows:

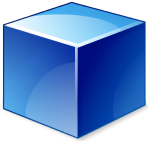
$$S^\circ = \{y : x \cdot y \leq 1 \text{ for all } x \in S\}$$

Note

- Every halfspace $a^\intercal x \leq b$ with $b \neq 0$ can be written as a "normalized" inequality $y^\intercal x \leq 1$, by dividing by b.
- S° can be thought of as the normalized representations of halfspaces containing S.

Properties of the Polar

1. If S is bounded and $0 \in interior(S)$, then the same holds for S°.
2. $S^{\circ\circ} = S$



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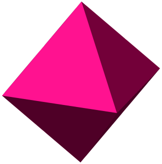

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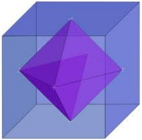
Polarity of Polytopes



Polytopes

Given a polytope P represented as $Ax \preceq \vec{1}$, the polar P° is the convex hull of the rows of A.

- Facets of P correspond to vertices of P°.
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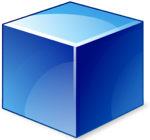
Proof Outline: Optimization $\Rightarrow$ Separation



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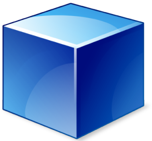

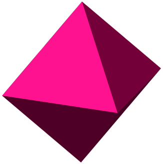

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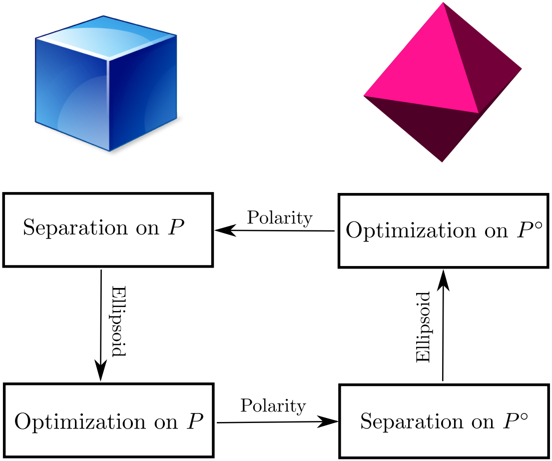

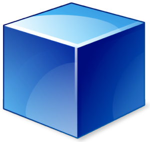

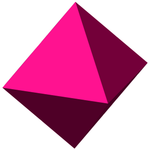

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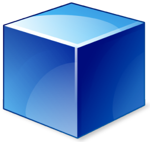

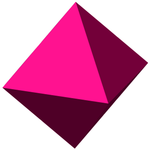

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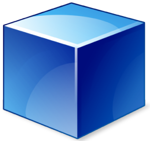
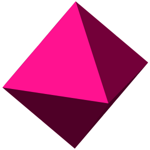
Separation over S reduces in constant time to optimization over S°, and vice versa since $S^{\circ\circ} = S$.





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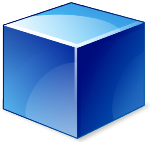

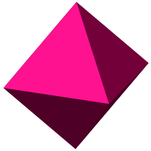

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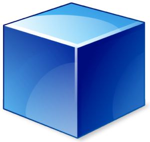 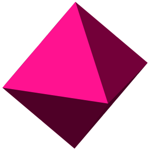

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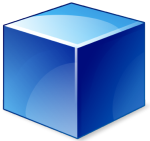

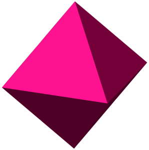

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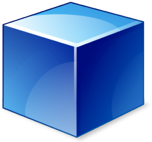
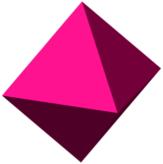
- If we find $y \in S^\circ$ s.t. $y \cdot x > 1$, then y is the separating hyperplane
    - $y^\intercal z \leq 1 < y^\intercal x$ for every $z \in S$.

Optimization $\iff$ Separation





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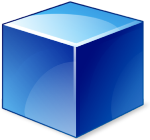 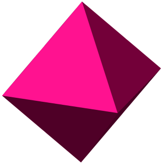

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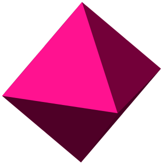
Technical Note 1

Need to “center” polytopes about origin. Can do that by running ellipsoid method to find a strictly feasible point in P.

Optimization $\Longleftrightarrow$ Separation



$$\begin{tikzcd}
\text{Separation on } P \arrow[d, "\text{Ellipsoid}"] & \text{Optimization on } P^\circ \arrow[l, "\text{Polarity}"'] \\
\text{Optimization on } P \arrow[r, "\text{Polarity}"] & \text{Separation on } P^\circ \arrow[u, "\text{Ellipsoid}"']
\end{tikzcd}$$

Technical Note 2

For up arrow (applying ellipsoid to P°), need polynomial bit complexity of facets of P°. Follows from polynomial bit complexity of vertices of P.

Beyond Polytopes

Essentially everything we proved about equivalence of separation and optimization for polytopes extends (approximately) to arbitrary convex sets, so long as you can circumscribe the convex set.

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Given closed convex $P \subseteq \mathbb{R}^n$, and radius R s.t. $P \subseteq B(0, R)$:

Weak Optimization Problem

- Input: Linear objective $c \in \mathbb{R}^n$.
- Output: $x \in P^{+\epsilon}$, and $c^\intercal x \geq \max_{x' \in P} c^\intercal x' - \epsilon$

Weak Separation Problem

- Input: $y \in \mathbb{R}^n$
- Output: Decide that $y \in P^{-\epsilon}$, or else find $h \in \mathbb{R}^n$ with $||h|| = 1$ s.t. $h^\intercal x < h^\intercal y + \epsilon$ for all $x \in P$.

Theorem (Equivalence of Separation and Optimization for Convex Sets)

Consider a family $\mathcal{P}$ *of convex sets described implicitly using* $\langle P \rangle$ *bits, and suppose that for each* $P \in \mathcal{P}$ *we are also given rational* R *s.t.* $P \subseteq B(0, R)$. *The weak separation problem is solvable in* $\text{poly}(\langle P \rangle, \langle R \rangle, n, \langle y \rangle, \log(1/\epsilon))$ *time for* $P \in \mathcal{P}$ *if and only if the weak optimization problem is solvable in* $\text{poly}(\langle P \rangle, \langle R \rangle, n, \langle c \rangle, \log(1/\epsilon))$ *time.*

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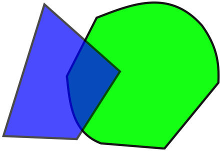
- By polarity, weak optimization is equivalent to weak separation
- For proof / details, see the GLS book.

Implication: Operations preserving solvability

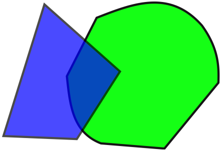


- Assume you can efficiently optimize over two convex sets P and Q

Question

What about $P \bigcap Q$ and $P \bigcup Q$?

Implication: Operations preserving solvability



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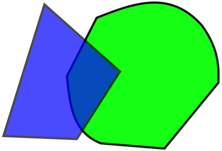
Question

What about $P \bigcap Q$ and $P \bigcup Q$?

$P \bigcap Q$

- Yes! Follows from equivalence of separation and optimization.
- Specifically, can separate over P and Q individually, therefore can separate over $P \bigcap Q$, and then can optimize over $P \bigcap Q$.
- Applications: colorful spanning tree, cardinality-constrained matching, ...

Implication: Operations preserving solvability



- Assume you can efficiently optimize over two convex sets P and Q

Question

What about $P \bigcap Q$ and $P \bigcup Q$?

$P \bigcup Q$

- Yes! Simply optimize over each separately, and take the better of the two outcomes.
- Equivalent to optimizing over the convex hull of $P \bigcup Q$.
- Implication of Separation/optimization equivalence: there is a separation oracle for $convexhull(P \bigcup Q)$.

Implication: Constructive Caratheodory

Problem

Given a point $x \in \mathcal{P}$, where $\mathcal{P} \subseteq \mathbb{R}^n$ is a solvable polytope, write x as a convex combination of $n+1$ vertices of $\mathcal{P}$, and do so in polynomial time.

- Existence: Caratheodory's theorem.
- E.g. Birkhoff Von-Neumann, fractional spanning trees, fractional matchings, . . .
- Follows from equivalence of separation and optimization. See HW.