Homework #2
CS675 Fall 2023

Due Wednesday Sept 6, by midnight

General Instructions The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems — doing so would be considered cheating.

Several of these problems are drawn from the following texts, each of which is linked on the course website: Luenberger and Ye (4th edition), Korte and Vygen (5th edition), and Boyd and Vendenberghe. Please make sure you are using the correct edition of each of the books by using the links on the course website.

We request that you submit your homework as a pdf file, by email to the TA.

Finally, whenever a question asks you to "show" or "prove" a claim, please provide a formal mathematical proof.

Problem 1. (5 points)
L&Y Chapter 2, Exercise 10.

Problem 2. (5 points)
Using the fundamental theorem of LP, prove the following finite version of Caratheodory's theorem: If X is a finite subset of $\mathbb{R}^n$, and y is in the convex hull of X, then y is a convex combination of at most $n+1$ points in X.
(Hint: Write an LP expressing y as a convex combination of points in X, then use our trick for counting the number of non-zero variables).

Problem 3. (10 points)
Given a matrix $A \in \mathbb{R}^{m \times n}$, show that exactly one of the following systems has a solution

- $Ax \succ 0$, $x \in \mathbb{R}^n$ (note: Ax is entry-wise *strictly* greater than zero)
- $A^\intercal y = 0$, $y \succeq 0$, and $y \in \mathbb{R}^m$ is non-zero

Problem 4. (10 points)
Recall the optimal production problem we introduced in class. We will now consider a generalization of that problem to a setting with multiple colluding firms. As in the optimal production problem, there are n products and m resources, where producing a unit of the j'th product consumes A_{ij}

units of the i'th resource, and each unit of the j'th product can be sold at a profit of c_j. There are K firms, the k'th of whom is endowed with b_i^k units of the i'th resource — we use $b^k \in \mathbb{R}_+^m$ to denote the endowment of the k'th firm. We allow a subset $S \subseteq \{1, \ldots, K\}$ of the firms to *collude*, in which case the firms pool their resources in order to maximize their collective profit, effectively solving the following optimization problem.

$$
\begin{array}{rll}
OPT(S) = & \text{maximize} & c^\intercal x \\
& \text{subject to} & Ax \preceq \sum_{k \in S} b^k \\
& & x \succeq 0
\end{array}
$$

We say a coalition of firms is *stable* if the firms can share profit in such a way so that no subset of the coalition can gain by breaking with the group and forming a coalition of their own. Formally, coalition $S \subseteq \{1, \ldots, K\}$ is stable if there are profit shares $p \in \mathbb{R}_+^S$ such that

1. $\sum_{k \in S} p_k = OPT(S)$. i.e. the total profit distributed equals the aggregate profit of the coalition.

2. $\sum_{k \in T} p_k \geq OPT(T)$ for all $T \subseteq S$. i.e. no subset of the coalition can collectively increase their profit by breaking off from S.

Show that the grand coalition $\{1, \ldots, K\}$ is stable.