Homework #7
CS675 Fall 2023

Due Wednesday Nov 1, by midnight

General Instructions The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems — doing so would be considered cheating.

Several of these problems are drawn from the following texts, each of which is linked on the course website: Luenberger and Ye (4th edition), Korte and Vygen (5th edition), and Boyd and Vendenberghe. Please make sure you are using the correct edition of each of the books by using the links on the course website.

We request that you submit your homework as a pdf file, by email to the TA.

Finally, whenever a question asks you to "show" or "prove" a claim, please provide a formal mathematical proof.

Problem 1. (15 points)
Consider the linear program concerned with minimizing $\langle c, x \rangle$ subject to linear constraints $Ax \succeq b$, and let $\mathrm{OPT}(c, b)$ be its optimal value as a function of vectors $\vec{c}$ and $\vec{b}$.

(a) [5 points]. Is OPT convex or concave with respect to b?

(b) [5 points]. Is OPT convex or concave with respect to c?

(c) [5 points]. How does your answer to parts (a) and (b) change if we change the linear program to maximize $\langle c, x \rangle$, or if we change the constraints to $Ax \preceq b$, or both?

Problem 2. (16 points)
In this problem, we will show how to implement Caratheodory's theorem efficiently, assuming only that the relevant extreme points form a solvable polytope. Formally, let $\mathcal{P} = \{x : Ax \preceq b\} \subseteq \mathbb{R}^n$ be a polytope given implicitly by a *separation oracle*: a procedure which takes as input $y \in \mathbb{R}^n$, and either certifies $y \in \mathcal{P}$ or else returns one of the constraints $Ax \preceq b$ which is violated at $x = y$. In the *contructive Caratheodory problem* for $\mathcal{P}$, we are given a point $y \in \mathbb{R}^n$, and we must output a representation of y as a convex combination of at most $n + 1$ vertices of $\mathcal{P}$, or else declare that $y \notin \mathcal{P}$. You may assume that $\mathcal{P}$ is full dimensional if this simplifies your proofs, though this is not necessary.

(a) [4 points]. Given $y \in \mathcal{P}$, show that there exist two points y_1 and y_2 on the boundary of $\mathcal{P}$, and $\alpha \in [0, 1]$, such that $y = \alpha y_1 + (1 - \alpha) y_2$. Show how to compute such a y_1, y_2, and α. The runtime of your algorithm should poly$(n, B, \langle y \rangle)$, plus poly$(n, B, \langle y \rangle)$ calls to the separation oracle, where B is an upperbound on the bit complexity of individual entries of A and b.

(b) [4 points]. Building on your solution to part (a), argue that you can also efficiently identify a constraint from the system $Ax \preceq b$ which is tight at y_1 (and similarly for y_2). The runtime of your algorithm should be as specified for part (a).
(**Hint:** When $y_1 \neq y_2$, you can compute a point y_1' just past y_1 on the line connecting y_2 to y_1 such that invoking the separation oracle for y_1' returns a constraint which is tight at y_1. When $y_1 = y_2$ a somewhat similar trick works. There are tedious numerical issues here which you are permitted to discuss informally.)

(c) [8 points]. Building on your solution to (a) and (b), describe a recursive algorithm which solves the constructive Caratheodory problem. The runtime of your algorithm should be as specified for part (a).

Problem 3. (16 points)
In class, we stated a quite powerful solvability result for implicitly-described linear programs. Specifically, given access to a separation oracle for the set $P = \{x \in \mathbb{R}^n : Ax \preceq b\}$, and the guarantee that every number occurring in $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ is a rational number with at most B bits in its binary representation, an optimal solution to the linear program $\max_{x \in P} c^\intercal x$ can be computed with poly(n, B) overhead in runtime. Crucially, the runtime guarantee is independent of m, the number of constraints of the (hidden) linear program, which may be very large (say, exponential or doubly exponential in n). In this problem, you will prove that a solution to a (normalized) dual linear program can be obtained as well, despite the fact that the number of variables in the dual LP is very large.

(a) [4 points]. Solving the dual LP is hopeless if we can't express the solution succinctly. Given a primal LP of the form described above, write down the dual LP, and explain how an optimal dual solution may be represented in poly(n, B) bits.

(b) [4 points]. Explain why, in general, given a linear program of the form $\max \{c^\intercal x : Ax \preceq b\}$ with the feasible set given by an arbitrary separation oracle, you could never hope to obtain the optimal dual solution, or any feasible dual solution for that matter.

(c) [8 points]. In light of (b), we will instead solve the dual of a "normalized" primal LP. Assume that $b \succ 0$.[1] Now consider the normalized primal LP given by $\max \left\{ c^\intercal x : A'x \preceq \vec{1} \right\}$ with $A'_{ij} = A_{ij}/b_i$, and note that it is equivalent to the original primal LP. Under the same assumptions described in the preamble, show how to obtain an optimal solution to the dual of the normalized primal with $\text{poly}(n, B)$ overhead in runtime.
(**Hint:** This should remind you of a problem on a previous homework. Use polars and your construction for Caratheodory's theorem from the previous problem.)

Problem 4. (9 points)
Let $\mathcal{X} = \{v_1, \dots, v_m\}$ where $v_i \in \mathbb{R}^n$ for $i = 1, \dots, m$. As explained in class, the set system over $\mathcal{X}$ whose feasible sets are the linearly independent subsets of $\mathcal{X}$ is a matroid. In this problem, we will explore whether other forms of independence also yield matroids.

(a) [3 points]. Suppose instead that we define $S \subseteq \mathcal{X}$ to be feasible if the vectors in S are independent with respect to convex combinations — i.e. no vector $v \in S$ can be written as a convex combination of the remaining vectors $S \setminus v$. Is the resulting set system a matroid? Prove or disprove.

(b) [3 points]. What if we define $S \subseteq \mathcal{X}$ to be feasible if the vectors in S are affinely independent — i.e. no vector $v \in S$ can be written as an affine combination of the remaining vectors $S \setminus v$. Is the resulting set system a matroid? Prove or disprove.

(c) [3 points]. Finally, what if we say $S \subseteq \mathcal{X}$ is feasible if the vectors in S are conically independent — i.e. no vector $v \in S$ can be written as a conic combination of the remaining vectors $S \setminus v$. Is the resulting set system a matroid? Prove or disprove.

[1] This is without loss of generality. You could use the ellipsoid method to compute a strictly feasible solution to the primal, then shift the origin so that it is strictly feasible, which would guarantee that $b \succ 0$. However, you do not have to prove any of this for this problem.